THE APRIL MEETING OF THE CHICAGO SECTION.

THE fifteenth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at Northwestern University, Evanston, Ill., on Saturday, April 2, 1904. The following members were present:

Dr. G. A. Bliss, Professor Oskar Bolza, Professor D. F. Campbell, Professor L. W. Dowling, Mr. E. B. Escott, Professor Arthur G. Hall, Dr. C. N. Haskins, Professor E. R. Hedrick, Professor Thomas F. Holgate, Dr. H. G. Keppel, Mr. N. J. Lennes, Mr. A. C. Lunn, Professor Heinrich Maschke, Professor E. H. Moore, Professor G. W. Myers, Dr. H. L. Rietz, Professor J. B. Shaw, Professor E. B. Skinner, Professor S. E. Slocum, Professor E. J. Townsend, Dr. Oswald Veblen, Professor C. A. Waldo, Professor H. S. White, Mr. J. W. Young, Professor Alexander Ziwet.

The first session opened at ten o'clock, Professor George W. Myers, of the University of Chicago, in the chair. The following papers were read:

(1) Dr. A. E. YOUNG: "Surfaces which have $D = \pm D''$ when referred to isothermal lines."

(2) Dr. G. A. BLISS: "On the sufficient condition for a minimum with respect to one-sided variations in the calculus of variations."

(3) Mr. A. C. LUNN: "Transformations of the equation of plane waves."

(4) Mr. E. B. ESCOTT: "Some new logarithmic series."

(5) Professor J. B. SHAW: "Group algebras defined by the groups whose generating equations are $e_1^n = 1 = e_2^r$, $e_1 e_2 = e_2 e_1^m$."

(6) Dr. SAUL EPSTEEN and Mr. H. B. LEONARD: "On the definition of reducible linear associative algebras."

(7) Professor J. B. SHAW: "The group determined by a matrix of order n is of order n^3."

(8) Mr. A. C. LUNN: "The fundamental theorems on ordinary differential equations in real variables."

(9) Mr. E. B. ESCOTT: "Expression of the square root as a continued fraction."

(10) Dr. G. A. BLISS: "The exterior and interior of a plane curve."

(11) Professor S. E. SLOCUM: "The strength of flat plates, with an application to concrete-steel floor panels."

(12) Dr. SAUL EPSTEEN: "On linear homogeneous difference equations."

(13) Professor W. J. RUSK: "Note on the nth derivative of a determinant whose constituents are functions of a given variable."

(14) Professor L. E. DICKSON: "Application of groups to a complex problem in arrangements."

(15) Professor L. E. DICKSON: "The abstract group G simply isomorphic with the group of linear fractional transformations in a Galois field."

Dr. Epsteen and Mr. Leonard were introduced by Professor Moore. In the absence of the authors the papers by Professor Rusk and Professor Dickson were read by title.

During the noon hour luncheon was served, at which over forty persons were present. This was followed by a discussion of the present state of mathematical study in Europe, participated in by Professor Maschke, Dr. Bliss, Professor Townsend and Professor Hedrick.

Abstracts of the papers presented follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Dr. A. E. Young considered the problem in differential geometry of determining those surfaces which have $D = \pm D''$, when referred to isothermal lines, and when the parameters chosen are such that the square of the linear element takes the form $\lambda(du^2 + dv^2)$, the D's being the well known quantities appearing in the second fundamental form. Considering first the case $D = -D''$, he finds that every surface whose fundamental forms satisfy these conditions, is a minimal surface. Also, all minimal surfaces satisfy these conditions. Finally, minimal surfaces are characterized by the property that when referred to any isothermal system whatsoever, they have $D = -D''$ in their second fundamental form.

Proceeding to the case where $D = D''$, Dr. Young finds, as a solution of the Codazzi and Gauss equations, surfaces whose linear element takes the form

$$ds^2 = \left[\frac{[1 - e^{a(u+v)}]^2}{be^{a(u+v)}}\right]^{\tan^2 w/2}(du^2 + dv^2),$$

where w is the asymptotic angle.

These surfaces, infinite in number, are shown to be peculiarly related to one another and to the catenoid. They all have a constant angle between their asymptotic lines, but no two the same angle.

Lastly, there is a system of surfaces for which $D = D''$ whose linear element takes the form

$$ds^2 = [f(u+v) + \varphi(u-v)]^{(c-1)/c}\,[du^2 + dv^2],$$

the possible values of c and the form of the functions f, φ being determined by the Gauss equation.

The paper closes with the complete determination of these latter surfaces.

2. In the usual problem of the calculus of variations, a curve E, $y = y(x)$, is sought which joins two fixed points p and q and gives to an integral

$$I = \int f(x, y, y')dx$$

a smaller value than any other curve C joining p and q. In Dr. Bliss's first paper the curves C are supposed to lie on one side only of the curve E, and a set of sufficient conditions for a minimum under those circumstances is derived. The result is of importance when the curves along which I is taken are restricted to lie within a region R of the (x, y)-plane. In that case an arc of the solution E may coincide with the boundary of R, and the only allowable variations of such an arc are therefore all on one side.

3. Mr. Lunn's note gave the form of the function group defined by the condition that the equation of plane waves be invariant. The analogy with the theory of conformal transformation was pointed out, and the method applied to the construction of classes of possible vibration modes whose computation by Fourier's series would be comparatively tedious.

4. In Mr. Escott's paper, the author starts with the well-known series

$$\log \frac{1+y}{1-y} = 2(y + \tfrac{1}{3}y^3 + \tfrac{1}{5}y^5 + \cdots).$$

If we replace y by $C(a_0x^n + a_1x^{n-1} \cdots + a_n)^{-1}$ the first member takes the form of a fraction, the numerator and denomi-

nator being polynomials of the nth degree and differing only in their absolute terms. The problem is to determine the coefficients so that the numerator and denominator may have rational, linear factors. The problem is completely solved for $n = 1$, 2, 3. For $n = 4, 5, 6$, an indefinite number of solutions may be found, some of which have been given before. For $n = 10$ four solutions have been discovered, two of the factors however being of the second degree. These are believed to be new. It is shown how the computation of logarithmic tables might be considerably abbreviated by the use of these series.

5. In his first paper, Professor Shaw deduces the conditions which groups of the class defined must satisfy, and develops the general form of corresponding group algebras. From these forms the characters of all operators of the group may be found easily.

6. A linear associative algebra $E_i \equiv E_j E_k \equiv e_1 \cdots e_m e_{m+1} \cdots e_n$, the general number being written $X = J + K$, is said to be reducible and it contains a modulus when the following seven conditions are fulfilled: A) associativity, *i. e.*, $(X_1X_2)X_3 = X_1(X_2X_3)$; C_1) $J_1J_2 = J_3$; C_2) $K_1K_2 = K_3$; C_{jk}) $JK = 0$; C_{kj}) $KJ = 0$; C_r) right hand division possible and unique; C_l) left hand division possible and unique. It has been shown (Epsteen, *Transactions*, January, 1904) that the four independent conditions A, C_{jk}, C_{kj}, C_r or C_l suffice for the definition of reducibility.

In the present paper of Dr. Epsteen and Mr. Leonard the above seven conditions are broken up into twenty constituent parts as follows: Since $i = j + k$, A is equivalent to eight conditions $A_1, \cdots, A_8$, which are typified by A_1) $(J_1J_2)J_3 = J_1(J_2J_3)$. The condition C_{jk} is equivalent to the simultaneous fulfillment of C^j_{jk}) $J_1K_1 \neq J_2$ and C^k_{jk}) $K_3J_3 \neq K_4$. Similarly C_{kj} is equivalent to C^j_{kj} and C^k_{kj}. Besides the above conditions C_1, C_2, C_r, C_l there are also included C^j_r, C^k_r; C^j_l, C^k_l.

The authors investigate (1) the dependencies among these twenty conditions, (2) all possible modes of defining reducible linear associative algebras, (3) independence proofs of the bodies of conditions which constitute the various definitions of reducibility.

7. In his second paper, Professor Shaw gives a simple proof of the theorem given in the title, deducing also the general form of group algebras defined by the group.

8. Mr. Lunn's second paper deals with sets of sufficient conditions for some of the fundamental theorems on ordinary differential equations in real variables. The theorem of continuity of the solutions as functions of parameters was given in the general case of parameters defined on any set with a cluster point, and shown to have as corollary an extension to differtial equations of the term by term integration of infinite series; this was illustrated by examples from the theory of resonance in vibrating systems, and by certain methods of approximation in celestial mechanics. The theorem of differentiability with respect to the parameters was reduced to the previous theorem and the results applied to equations involving holomorphic functions, and to the commutativity of transformations defined by their infinitesimal transformations. The point transformations defined by a system of differential equations were considered with respect to the maintenance of properties of sets of points, such as continuity, tangency, measurability, with application to the theorems of hydrodynamics.

9. Mr. Escott's second paper considers the relation between a number N and the partial quotients obtained in expressing $\sqrt{N}$ as a continued fraction. A number of special forms of N were given by Euler, Degen, Stern, and others, for which the continued fraction could be written down at once. In this paper a general formula is given which includes all cases. These formulas are expressed by means of Euler's function $[a, b, \cdots m, n]$ (Muir's continuant).

Among the results obtained is this theorem: If $N = 4n + 3$ is a prime number, the continued fraction has a single middle term, a or $a - 1$ according as a is odd or even, a being the greatest integer in $\sqrt{N}$.

10. The theorem that a continuous closed curve divides the plane into an interior and an exterior has been discussed by Jordan, Schoenflies, and in a recent number of the BULLETIN by Ames. The two former assume the theorem for polygons and then extend it to apply to more general curves. The proof of Ames applies to all the so-called regular curves, including polygons. In Dr. Bliss's second paper the theorem is proved by a different method for another class of curves which do not necessarily have a tangent. Among them are included all the curves considered by Schonflies, all curves consisting of a finite

number of analytic pieces, and all the regular curves for which a direction exists parallel to only a finite number of tangents.

11. In Dr. Slocum's paper a rational theory of stress distribution in flat, rectangular plates, fixed at the edges, is developed, and the results obtained are applied to the case of a concrete-steel floor panel in actual use.

To find what proportion of the load at any point of a horizontal plate is carried by each pair of opposite edges, the plate is supposed to be divided by vertical planes into narrow strips parallel to the edges. If we consider two such mutually perpendicular strips through any point of the plate and bearing a unit load at their intersection, the amount of this load transmitted by each strip to its respective abutments is inversely proportional to the deflection of the strip under such load. In this way the equation giving the distribution of load along any strip is obtained and found to be a septic, and from this the position of maximum and minimum load is determined.

In order to simplify the problem sufficiently for practical use, it is then shown that throughout the region under discussion the septic may be replaced by a parabola without involving an error greater than .001 of the total load. Since the maximum bending stress occurs at the edges of the plate, in finding this stress it is allowable to consider any particular strip as an independent beam and analyze it in the usual way. Under parabolic loading the distribution of stress in a strip of unit width is thus determined, and a simple formula for maximum stress deduced.

The case in which the supports are flexible is then taken up. This condition is usually present in the structural use of flat plates, such, for instance, as a concrete-steel floor panel monolithic with its supporting girders. After showing that this problem is indeterminate, an approximate solution is given and a limiting value of the maximum stress determined. The results of the preceding analysis are then applied to the calculation of the maximum stresses in a concrete-steel floor panel of recent design.

12. In the *Comptes Rendus* of October 27, 1903, Guldberg shows that to a linear homogeneous difference equation

$$E: \qquad y_{x+n} + p_x^{(1)} y_{x+n-1} + \cdots + p_x^{(n)} y_x = 0$$

there corresponds a linear homogeneous group

$$\Gamma: \qquad \overline{y}_x^{(i)} = \sum_{j=1}^{n} a_{ij} y_x^{(j)} \qquad (i = 1, \cdots, n)$$

which has the following double property: Every rational function of

$$x, \quad y_x^{(1)}, \quad \cdots \quad y_x^{(n)},$$

and their successive values (the y_x forming a fundamental system), which is expressible rationally as a function of x, remains invariant when the y's are transformed by Γ, and conversely.

Dr. Epsteen points out in his second paper that the above double theorem relates to *formal* and not to *numerical* invariance; he indicates the importance of specifying definitely a domain of rationality and then shows that the group of rationality G of E is a mixed group of r systems of linear homogeneous transformations

$$G: \qquad \overline{y}^{(i,j)} = \sum_{k=1}^{n} a_{ikj} y_x^{(k,j)} \qquad \left(\begin{smallmatrix} i=1, \cdots, n \\ j=0, 1 \cdots, r-1 \end{smallmatrix}\right).$$

By the adjunction of the solutions of an algebraic equation with rational coefficients to the domain of rationality, the group of rationality G is reduced to a continuous group G_1. When G is the identical transformation the difference equation is algebraically integrable.

13. Professor Rusk obtained a general expression for the nth derivative of a determinant.

14. The first paper by Professor Dickson determines all the group solutions of the following problem: Required $\frac{1}{2}(m-1)(m-2)$ arrangements of m persons at a round table such that no one shall be twice between the same two companions. Give to the arrangements the notation $AB...M$, the first letter being always A. We require that the substitutions by which they are obtained from a standard order $ABC...K$, together with their left-hand products by T which replaces $ABC...K$ by $AK...CB$, shall form a doubly transitive group G. Since G is of order $(m-1)(m-2)$ on $m-1$ letters, we must have $m-1 = p^n$, p being prime. Furthermore, G must be the group of linear transformations $x' = ax + b$ in the $GF[p^n]$, there being, however, a second group H_{72} for $p^n = 3^2$. For the first non-trivial case $p^n = 2^2$, the only solution is

$$\begin{array}{lllll} \infty & 0 & 1 & i & i^2, \\ \infty & 0 & i & i^2 & 1, \\ \infty & 0 & i^2 & 1 & i, \end{array} \qquad \begin{array}{lllll} \infty & 1 & 0 & i^2 & i, \\ \infty & 1 & i & 0 & i^2, \\ \infty & i & 0 & 1 & i^2. \end{array}$$

For $p^n = 5$, the only group solution is one equivalent to that by C. H. Judson in the *American Mathematical Monthly*, 1900, page 72. For $p^n = 7$ there are exactly four group solutions; for $p^n = 9$, there are four. The second group H_{72} actually furnishes solutions.

15. The second paper by Professor Dickson relates to the question of redundancies in Professor Moore's set of relations defining G for the $GF[p^n]$. It is an addition to a paper with the same title in the *Proceedings of the London Mathematical Society*, volume 35 (1902), pages 292–305, which treated the cases for which $p^n \leqq 48$. In the present paper, the theorem of the paper cited is proved, by fortunate devices, to hold for the two new cases $p^n = 3^5$ and $p^n = 5^3$.

EVANSTON, ILL.,
April 15, 1904.

THOMAS F. HOLGATE,
Secretary of the Section.

THE HEINE–BOREL THEOREM.

BY DR. OSWALD VEBLEN.

This note has for its main object a proof that the Heine-Borel theorem is equivalent, as a continuity axiom, to the Dedekind cut proposition.* In §3 the equivalence of the closure of a set with what may be called the H-B property is shown to apply not only to the continuum but to any set of numbers.† This result applies in manifolds of any number of dimensions as well as to linear sets.

1. I. *If every number of an interval AB belongs to at least one segment σ of a set of segments $[\sigma]$, then there exists a finite sub-*

* The equivalence in question suggested itself to Mr. N. J. Lennes and myself while we were working over some elementary propositions in real function theory.

† This extends a theorem of Dr. W. H. Young to the effect that every closed linear set has the H-B property. Cf. W. H. Young, "Overlapping Intervals." *Proceedings of the London Mathematical Society*, p. 384, vol. 35 (1903).

set of $[\sigma]$, $\sigma_1, \sigma_2, \sigma_3, \cdots, \sigma_n$, *such that every number of* AB *belongs to at least one of the segments* $\sigma_1, \sigma_2, \cdots \sigma_n$.

By a *segment* $AB = \sigma$ $(A < B)$ is meant the set of real numbers $[X]$ such that $A < X < B$. By an *interval* AB is meant the set of real numbers $[X]$ such that $A \leqq X \leqq B$. Proposition I is essentially involved in Heine's * proof of the theorem of uniform continuity, but has first been stated in explicit form by Borel, † who was careful to specify that the set of segments of which he spoke should be at most numerably infinite. The theorem, however, is valid whatever be the set $[\sigma]$, and indeed to the restricted form given it by Borel the remarks below do not wholly apply.

In its generalized form, the Heine-Borel theorem can be used to replace the tedious "caging" process which appears so frequently in rigorous analysis. As an example, let us take Weierstrass's theorem that any infinite set of numbers $[X]$ lying on a bounded interval AB has at least one limit number. If the theorem should fail, then about each number of the interval AB there would be a segment σ containing not more than one number of the set $[X]$. By the Heine-Borel theorem, there would exist a finite subset $\sigma_1, \sigma_2, \cdots, \sigma_n$ of these segments σ such that every number of AB would lie on at least one such segment. But as each of $\sigma_1, \sigma_2, \cdots, \sigma_n$ contained only one number of $[X]$, $[X]$ could contain only a finite number of numbers, contrary to hypothesis.

2. Besides being of great convenience in the routine of limiting processes, the generalized form of the Heine-Borel theorem is equally fundamental with the Dedekind cut proposition as an axiom of continuity for the real number system. The Dedekind statement is as follows:

II. *If all the numbers of an interval* AB *fall into two classes* $[M]$ *and* $[N]$, *such that every* M *is greater than every* N, *then there exists a number* O *such that* $M \geqq O \geqq N$ *for every* M *and every* N.

The assertion that I and II are equally fundamental may be more exactly stated as follows:

III. *Admitting the ordinal relations of the real number system,* I *is a consequence of* II *and* II *is a consequence of* I.

* Heine, "Die Elemente der Functionenlehre," *Crelle*, vol. 74 (1872), p. 188.

† Borel, "Sur quelques points de la théorie des fonctions," *Annales de l'École Normale Supérieure*, 3d series, vol. 12 (1895), p. 51.

The proof that I is a consequence of II is well-known* and is given in a generalized form below. To prove II from I, observe that if there is no number O of the kind required, $[M]$ has no greatest lower bound and $[N]$ no least upper bound. Therefore, about each number M there is a segment not including any number N, and about each number N a segment not including any number M. Every number of AB being either an M or an N, this set of segments satisfies the hypothesis of I. Hence there is a finite set of segments $\sigma_1, \sigma_2, \cdots, \sigma_n$ including every number of AB and each including only numbers of $[M]$ or only numbers of $[N]$. Let A be the smaller of A and B. Of the finite number of segments which include A, no one includes or is bounded by a number of $[M]$. Let the largest bound of the segments including A be A'. Proceed with A' (which is a number of $[N]$) as with A, obtaining A'' of $[N]$ and so on. Since there are a finite number of segments $\sigma_1, \cdots, \sigma_n$, the process leads by a finite number of steps to the conclusion that none of the segments include numbers of $[M]$, contrary to hypothesis.

3. The Heine-Borel theorem has a generalization of some importance in the theory of sets. Let us say that a set of real numbers $[X]$ has the H-B property if, from every set of segments $[\sigma]$ such that every number X belongs to at least one σ, there can be selected a finite subset $\sigma_1, \sigma_2, \cdots, \sigma_n$ such that every X belongs to at least one σ_k $(k = 1, 2, \cdots, n)$. By a *closed* set, let us mean one that includes all its limit numbers.

IV. *A necessary and sufficient condition that a bounded set* $[X]$ *of numbers shall have the H-B property is that* $[X]$ *is closed.*

It is evident that an infinite set not including its limit numbers has not the H-B property. For if $[X]$ lies on AB $(A<B)$ and has a limit number C, not an X, let $[\underline{C}]$ be the set of all numbers smaller than C, and let $[\overline{C}]$ be the set of all numbers larger than C. The set of segments, $A\underline{C}$, $B\overline{C}$ has no finite subset including every number X.

In proving that a closed set has the H-B property, we assume without loss of generality that $A < B$ and that A and B belong to $[X]$. Either, one of the segments $\sigma = A_0A_1$ includes AB, in which case $n = 1$, or the ends $[A']$ of the seg-

* Cf. Schoenflies, "Bericht über die Mengenlehre," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 8 (1899), p. 51.

ments including A are all less than B. It is an immediate and well-known consequence of II that every bounded set of numbers has a least upper bound. Let A_1 be the least upper bound of $[A']$. Either A_1 belongs to the set $[X]$ or, by the closure of $[X]$, one of the segments including A includes all numbers X between A and A_1. If A_1 is an X, we denote it by X_1, if not, let X_1 be the smallest number X (existent because $[X]$ is closed) which is greater than A_1. Proceed with X_1 as with A_1, obtaining A_2 and, if necessary, X_2. By a finite number of steps, we obtain $A_k \geqq B$; for, if not, the bounded set $X_1, X_2, \cdots$ would have a least upper bound X_ω which, because of the closure of $[X]$, would be a limit number of $[X]$. Hence there would be a segment σ including X_ω; this segment must include at least one X_m and, therefore A_{m+1}, contrary to the hypothesis that A_{m+1} is the least upper bound of the ends of the segments including X_m.

To obtain the set $\sigma_1, \cdots, \sigma_n$, let $\sigma_1, \cdots, \sigma_k$ be a set of segments of $[\sigma]$ including $X_1, X_2, \cdots, X_{k-1}, B$; let σ_{k+1} be the segment of $[\sigma]$ including A and all numbers of $[X]$ between A and X_1 in case $A_1 \neq X_1$ and a segment of $[\sigma]$ including A and the lower end of σ_1 in case $A_1 = X_1$; similarly, let $\sigma_{k+h} (h = 2, \cdots, k)$ include X_{h-1} and all numbers between X_{h-1} and X_h if $X_h \neq A_h$ or include X_{h-1} and the lower end of σ_h if $A_h = X_h$. $n = 2k$ (unless $n = 1$ as above).

4. The theorems proved above depend upon the linear arrangement and continuity of the real number system and not at all upon the operations of addition and multiplication. They therefore apply equally well in the geometry of a straight line, even without metric hypotheses, and in fact to any interval of a continuous curve without multiple points.

The validity of the Heine-Borel theorem for the plane (or for space of any number of dimensions) suggests that it can be used as a continuity axiom, not only for real numbers, but also for complex number systems.

THE UNIVERSITY OF CHICAGO,
March 26, 1904.

ON SELF-DUAL SCROLLS.

BY MR. C. H. SISAM.

(Read before the American Mathematical Society, April 30, 1904.)

THEOREM. *The necessary and sufficient condition that a scroll be self-dual is that it belong to a non-special linear complex.*

That the above condition is sufficient was stated, without proof, by Lie in the *Mathematische Annalen*, volume 5, page 179. Last year Professor Wilczynski proved the theorem by the use of differential equations. His proof is published in the *Mathematische Annalen*, volume 58, page 249. The following proof is algebraic.

Let the scroll belong to a non-special linear complex. The latter may be expressed in the form

$$p_{12} = \kappa p_{34} \qquad (\kappa \neq 0).$$

The equations of the scroll may be written

$$x_i = a_i(\lambda) + \mu a_i'(\lambda) \qquad (a_1' = a_4 = 0),$$

λ, μ being the Gaussian coördinates. But $p_{12} = a_1 a_2', p_{34} = a_3 a_4'$, hence $a_1 a_2' = \kappa a_3 a_4'$ and we may replace the preceding equations by

$$x_1 = a_1(\lambda), \qquad x_2 = a_2(\lambda) + \mu a_3(\lambda),$$
$$x_3 = a_3(\lambda) + \mu a_3''(\lambda), \qquad x_4 = \mu\kappa a_1(\lambda).$$

This scroll may be defined as the locus of the intersection of corresponding planes of the cones

$$(1) \quad \kappa a_3 x_1 - a_1 \kappa x_3 + a_3'' x_4 = 0, \quad \kappa a_2 x_1 - a_1 \kappa x_2 + a_3 x_4 = 0.$$

The dual scroll is developed by the cones

$$(2) \quad a_1 x_1 + a_2 x_2 + a_3 x_3 = 0, \quad a_3 x_2 + a_3'' x_3 + \kappa a_1 x_4 = 0.$$

If we perform on (2) the operation

$$x_1 = -\kappa x_2, \quad x_2 = \kappa x_1, \quad x_3 = x_4, \quad x_4 = -x_3,$$

it becomes identical with (1), hence by this projection, the dual scroll goes into the original one. A scroll which belongs to a non-special linear complex is therefore self-dual.

Conversely, let the scroll whose equations are

$$x_i = a_i(\lambda) + \mu a_i'(\lambda) \qquad (i = 1, 2, 3, 4) \tag{3}$$

be self-dual. The equations of the dual of (3) may be written

$$\sum_{i=1}^{4} a_i x_i = 0, \qquad \sum_{i=1}^{4} a_i' x_i = 0. \tag{4}$$

Since (3) is self-dual (4) can be projected into (3). By putting

$$x_i = \sum_{j=1}^{4} \alpha_{ij} x_j \qquad (i = 1, 2, 3, 4),$$

(4) becomes

$$\sum_{i,j=1}^{4} \alpha_{ij} a_i x_j = 0, \qquad \sum_{i,j=1}^{4} \alpha_{ij} a_i' x_j = 0. \tag{5}$$

Since (3) and (5) represent the same scroll, we must have

$$\sum_{i,j=1}^{4} \alpha_{ij} a_i (a_i + \mu a_j') = 0, \qquad \sum_{i,j=1}^{4} \alpha_{ij} a_i' (a_i + \mu a_j') = 0,$$

or, since μ is arbitrary, we obtain

$$\Sigma \alpha_{ij} a_i a_j = 0, \ \Sigma \alpha_{ij} a_i a_j' = 0, \ \Sigma \alpha_{ij} a_i' a_j = 0, \ \Sigma \alpha_{ij} a_i' a_j' = 0. \tag{6}$$

The curves $x_i = a_i$ and $x_i = a_i'$ may obviously be taken so that they do not lie on the same quadric. The first and last of equations (6) must then be satisfied identically, hence

$$\alpha_{ij} + \alpha_{ji} = 0, \qquad (i, j = 1, 2, 3, 4).$$

From the second and third of equations (6) we then have

$$\Sigma \alpha_{ij} (a_i a_j' - \alpha_j a_i') = \Sigma \alpha_{ij} p_{ij} = 0,$$

which states that the scroll belongs to a linear complex.

Since the determinant of the transformation of coördinates is $(\alpha_{12}\alpha_{34} + \alpha_{13}\alpha_{42} + \alpha_{14}\alpha_{23})^2$, the complex is not special. Every self-dual scroll, therefore, belongs to a non-special linear complex.

CORNELL UNIVERSITY,
March 3, 1904.

ON SOME TENDENCIES IN GEOMETRIC INVESTIGATIONS.*

REMARKS ADDRESSED TO MY STUDENTS.†

By Professor Corrado Segre.

I.

Chasles at the end of his Aperçu historique says: "Dans la géométrie ancienne les vérités étaient isolées; de nouvelles étaient difficiles à imaginer, à créer; et ne devenait pas géomètre inventeur qui voulait.

"Aujourd'hui chacun peut se présenter, prendre une vérité quelconque connue, et la soumettre aux divers principes généraux de transformation; il en retirera d'autres vérités, différentes ou plus générales; et celles-ci seront susceptibles de pareilles opérations; de sorte qu'on pourra multiplier, presque à l'infini, le nombre des verités nouvelles déduites de la première: toutes, il est vrai, ne mériteront pas de voir le jour, mais un certain nombre d'entre elles pourront offrir de l'intérêt et conduire même à quelque chose de très-général.

"Peut donc qui voudra, dans l'état actuel de la science, généraliser et créer en géométrie; le génie n'est plus indispensable pour ajouter une pierre à l'édifice."

A half-century has passed, and during this period geometry has made immense progress.‡ The study, projectively, of

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*This article was originally published in the *Rivista di Matematica*, vol. 1 (1891), pp. 42–66, and has been translated for the Bulletin by Dr. J. W. Young. The translation has been made from a copy very kindly revised by the author, who has also approved the translation in manuscript. In a note to the translator Professor Segre says: "I have made only very slight changes in the text and a few additions in the foot-notes, chiefly of a bibliographic nature. The additions are all enclosed in [ ]."

†Yielding to the kind persistence of the editor of the *Rivista di Matematica*, I repeat here, connectedly and with some additions, certain considerations which from time to time I have had occasion to present at the university to students specializing in mathematics, and which especially aim to caution those who wish to devote themselves to scientific research against certain faults and errors into which young investigators — and particularly young geometers — easily fall. Such remarks will not seem inappropriate at the present time, when many young men in Italy are engaged with the study of geometry. But by their nature and scope, *they can present interest and novelty only to beginners: to them only may this paper be of some value.*

‡To get an approximate idea of the wealth of research which has been

algebraic curves and surfaces of the lowest orders was followed by the general theory of curves and surfaces of any order; with the extension of the concept of geometric *elements* came that of *algebraic manifolds* or *spreads* and of the kinds of *space* in which they are considered. New types of problems have presented themselves. And the projective transformations to which principally Chasles alluded have found their generalization in the infinitely broader conception of the *algebraic transformations*.

Thus, owing to the exceeding growth of geometric research as well as of methods of transformation, grew also proportionately the facility mentioned by the great French geometer of increasing without limit the number of new propositions, of generalizing and creating in geometry. And this facility which, at least apparently, is greater in this science than in analysis, in mathematical physics, etc., induces many young men, especially in Italy, to give geometry the preference over other branches of mathematics, some of which indeed are very sadly neglected by our students.

But facility is a bad counsellor; and often the work to which it leads the beginner, while it may serve as training, as preparation for original research, will not deserve to see the light. In the innumerable multitude of scientific publications geometric writings are not rare in which one would seek in vain for an idea at all novel, for a result which sooner or later might be of service, for anything in fact which might be destined to survive in the science; and one finds instead treatises on trivial problems or investigations on special forms which have absolutely no use, no importance, which have their origin not in the science itself but purely in the caprice of the author; or one finds applications of known methods which have already been made thousands of times; or generalizations from known results which are so easily made that the knowledge of the latter suffices to give at once the former; etc. Now such work is not merely useless; it is actually harmful because it produces a real incumbrance in the science and an embarrassment for more serious investigators; and because often it crowds out certain lines of thought which might well

made in this period the student need only glance at the historical monograph by Loria: "Il passato e il presente delle principali teorie geometriche," *Memorie della R. Accademia delle Scienze di Torino*, 2d series, vol. 38. [2d edition, Turin, 1896; German translation by Fritz Schütte, Leipzig, 1888.]

have deserved to be studied. Better, far better, that the student, instead of producing rapidly a long series of papers of such a nature, should work hard for a long time on the solution of a single problem, provided it is important: better one result fit to live than a thousand doomed to die at birth ! *

II.

But when is a question *important?* When does it deserve to be made the object of study?

It is impossible to give a precise answer to this question. The importance of a result is largely relative, is judged differently by different men, and changes with the times and circumstances. It has often happened that great importance has been attached to a problem merely on account of the difficulties which it has presented; and indeed if for its solution it has been necessary to invent new methods, noteworthy artifices, etc., the science has gained more perhaps through these than through the final result. In general we may call important all investigations relating to things which in themselves are important; all those which have a large degree of generality, or which unite under a single point of view subjects apparently distinct, simplifying or elucidating them; all those which lead to results that promise to be the source of numerous consequences; etc.

The study of the great masters is perhaps the best thing to recommend to the student who wishes to prepare himself to judge of the importance of problems. For it is precisely in the choice of these that the great minds have always shown themselves masters; and even when they have taken up very special problems, they have shown in what way those could become important. And here, in corroboration of the preceding, I will quote the words of Beltrami: † "Students should learn to study at an early stage the great works of the great masters instead

* To students who are looking toward the doctorate in mathematics it is well to say frankly that science should not be thought of as a profession in which all can succeed; though it be true that genius is no longer essential to produce useful results, still a certain aptitude is necessary; and he who knows himself to be without it should, with that veneration and sacrifice which science demands, renounce scientific research. Why should a young man, who could perhaps teach successfully the elementary mathematics and study thoroughly the numerous and important pedagogical questions which present themselves in his teaching, neglect such studies, in order to take up researches in higher mathematics which are not adapted to his type of mind?

† *Giornale di matematiche*, vol. 11, p. 153.

of making their minds sterile through the everlasting exercises of college, which are of no use whatever, except to produce a new Arcadia where indolence is veiled under the form of useless activity. . . . Hard study on the great models has ever brought out the strong; and of such must be our new scientific generation if it is to be worthy of the era to which it is born and of the struggles to which it is destined."

In such studies one should ever keep before him this other object: *to broaden as much as possible his own knowledge.* He who is interested only in works relating to the limited field which he is studying, will in the end give undue weight to questions which do not seem so important to another, who with broader knowledge looks at the subject from a higher point of view. It should be the aim of the student by an extended study of the best works in all branches, to attain a wider horizon with regard to the whole science.

III.

In a letter of Jacobi to Legendre (July 2, 1830) concerning a report by Poisson upon the Fundamenta Nova, we find:

". . . M. Poisson n'aurait pas dû reproduire dans son rapport une phrase peu adroite de feu M. Fourier, où ce dernier nous fait des reproches, à Abel et à moi, de ne pas nous être occupés de préférence du mouvement de la chaleur. Il est vrai que M. Fourier avait l'opinion que le but principal des mathématiques était l'utilité publique et l'explication des phénomènes naturels; mais un philosophe comme lui aurait dû savoir que le but unique de la science, c'est l' honneur de l' esprit humain, et que sous ce titre, une question de nombres vaut autant qu'une question du système du monde. . ."

There should be no doubt in regard to this view of Jacobi: *Science should be given absolutely the greatest liberty;* and in particular, we can not require it always to keep in sight the practical applications.

On the other hand, in the existing state of mathematical research in Italy, now that among our young mathematicians only very few are engaged in the study of mechanics, mathematical physics, etc., it is proper to recall the following very true words, which Fourier himself wrote eight years before in the preface of the Théorie analytique de la chaleur:

"L'étude approfondie de la nature est la source la plus féconde des découvertes mathématiques. Non seulement cette

étude, en offrant aux recherches un but déterminé, a l'avantage d'exclure les questions vagues et les calculs sans issue: elle est encore un moyen assuré de former l'analyse elle-même, et d'en découvrir les éléments qu'il nous importe le plus de connaître, et que cette science doit toujours conserver: ces éléments fondamentaux sont ceux qui se reproduisent dans tous les effets naturels."

Is it necessary to give examples in confirmation of the above; to record how the greater part of mathematical problems, from the simplest to the most advanced, have had their origin in applications to nature; how the theories of ordinary and partial differential equations, the trigonometric series, various new functions which later became of great importance in analysis, all originated in or received continual stimulation from mathematical physics or celestial mechanics; how the theory of the potential (and hence the subject of electrodynamics, etc.) has been united in a remarkable way with the theory of functions of a complex variable, and in particular of algebraic functions and their integrals, chiefly through the work of Riemann (that worthy successor of Gauss and Dirichlet) and his followers, so that one of these, Klein,* was able to refer to certain electrical phenomena to show the existence of the various abelian integrals? Or shall we recall how often the study of conics and quadrics has developed from that of natural phenomena; or how the null-system and the linear complex and congruence of straight lines, etc., have presented themselves in the mechanics of rigid bodies, while mechanics in turn has received extraordinary help from the geometry of straight lines (thanks chiefly to Plücker, Klein and Ball†). To give even a faint idea of the many different ways in which the problems of nature have spurred on the mathematical sciences to further progress, would be impossible; equally impossible would it be to enumerate all those great men, some of whom are still living and laboring, who have known how to combine the most advanced researches in pure mathematics with their applications to physics, astronomy, engineering.

Students who wish to devote themselves to scientific re-

* Ueber Riemanns Theorie der algebraischen Functionen und ihrer Integrale (Leipzig, 1882).

† The Theory of Screws (Dublin, 1876) [and A Treatise on the Theory of Screws (Cambridge, 1900)]. See also the work of Gravelius: Theoretische Mechanik starrer Systeme auf Grund der Methoden und Arbeiten Balls (Berlin, 1889).

search should keep before them these facts and these examples, and should study the pure theories at the same time with their applications. Among the many advantages which they will derive therefrom will always be this, that when they are without any other means of judging of the importance of a certain theoretical problem, there will ever remain the one furnished by its possible applications (which, as I have already said, though not *necessary*, is certainly *sufficient* to justify a scientific investigation).

IV.

What has been said of the relation between pure and applied mathematics applies to a far greater degree to the two principal subdivisions of pure mathematics, analysis and geometry. The method of coördinates serves to pass from one to the other and unites them intimately, or rather so welds them together that we may say that every advance in the one means an advance in the other. The whole of differential geometry stands in proof of this; in particular the manifold relations which it establishes between differential equations and curves, surfaces, the connexes introduced by Clebsch,* etc.; and the theory of continuous groups of transformations recently created by Lie has not only rendered great service to the theory of differential equations, but has on the other hand solved noteworthy geometrical problems, throwing light, for example, on the foundations of geometry.† The geometry of *algebraic* manifolds coincides, as is shown by the very name and definition of these forms,‡ with the analysis of algebraic functions and the related transcendental functions: thus projective geometry is the equivalent of the modern algebra of linear transformations (of invariants), etc.

* Concerning the researches of this great geometer on the connexes his biographers say (*Math. Annalen*, vol. 7, p. 50). "Er hat mit ihnen dem Grundzuge seiner mathematischen Denkweise noch einmal Ausdruck gegeben, welche *die Mathematik nicht als eine Reihe geschiedener, einander fremder Disciplinen, sondern als einen lebendigen Organismus erfassen wollte.*"

† Cf. Poincaré, "Sur les hypothèses fondamentales de la géométrie," *Bull. Soc. Math. de France*, vol. 15 (1887); and especially Lie, "Ueber die Grundlagen der Geometrie," *Berichte der k. sächs. Ges. d. W.*, Leipzig, 1890. [Lie, Theorie der Transformationsgruppen, III. Abschnitt 5, 1893.]

‡ A manifold or spread is said to be *algebraic* when its elements are the totality of those whose coördinates satisfy given algebraic equations (in which also indeterminate parameters may appear rationally). With this definition we may also include the notion of algebraic correspondence. [Cf. my Introduzione alla geometria sopra un ente algebrico semplicemente infinito, *Annali di Matematica* 2nd ser., vol. 22 (1893).]

And the theory of algebraic equations and of groups of substitutions throws light on many important problems of geometry, and in turn derives from the latter many useful representations and suggestions.* Etc.†

This great variety of bonds between analysis and geometry and the resulting necessity of studying them both together, and of not confining oneself to one alone of these two directions, the analytic or the synthetic, has always been recognized by the great mathematicians. Both Lagrange and Monge, for example, emphasized this in their courses in analysis and geometry at the Ecole normale (1795); and the latter in his Géométrie descriptive wrote the following, which evidently applies to the whole of geometry:

"... Ce n' est pas sans objet que nous comparons ici la géométrie descriptive et l'algèbre; ces deux sciences ont les rapports les plus intimes. Il n'y a aucune construction de géométrie descriptive qui ne puisse être traduite en analyse; et lorsque les questions ne comportent pas plus de trois inconnues,‡ chaque opération analytique peut être regardée comme l'écriture d'un spectacle en géométrie.

"*Il serait à désirer que ces deux sciences fussent cultivées ensemble*; la géométrie descriptive porterait dans les opérations analytiques les plus compliquées, l'évidence qui est son caractère; et, à son tour, l'analyse porterait dans la géométrie la généralité qui lui est propre."

And in more recent times Clebsch wrote regarding these two branches:§

"*Beide zusammen umfassen erst in Verein und Ergänzung das Ganze mathematischer Forschung, und es vermag keine von beiden auf die Dauer ohne schwere Schädigung ihres eigensten Wesens die Begleitung und den Einfluss der andern zu entbehren.*"

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* Cf. for example the chapter on geometrical applications in the Traité des substitutions et des équations algébriques of C. Jordan.

† We recall also Klein's Programm: Vergleichende Betrachtungen über neuere geometrische Forschungen (Erlangen, 1872) [reprinted in *Math. Annalen*, vol. 43; translated into Italian by G. Fano, *Annali di Matematica*, 2d series, vol. 17, and into English by M. W. Haskell, *Bulletin of the New York Math. Society*, vol. 2 (1892), p. 215].

‡ This restriction may be removed by the use of spreads of n dimensions. I shall return to this later.

§ "Zum Gedächtniss an Julius Plücker," *Abhandl. der k. Ges. d. W. zu Göttingen*, vol. 16 (1871) [reprinted in Plücker, Gesammelte mathematische Abhandlungen, Leipzig, 1895] translated into Italian by Professor Beltrami, *Giornale di Matematiche*, vol. 11. Compare this extract with that from the biography of Clebsch cited in a preceding note.

The same ideas are being advanced in lectures and by example by one of the greatest masters that Germany now boasts, Klein; indeed the whole school of which he is the head proves the advantage of possessing at once a knowledge of both analytic and geometric methods.

In Italy, however, the conditions are only too different.* The division of pure mathematics into analysis and geometry is made by young mathematicians in such a precise way† that it can be said of only a few that they are studying both. We have young analysts and young geometers; but young men who consider the whole of mathematics as their proper field, following the great examples that we have even in Italy, are very few. For this reason when I turn to my students of geometry, I feel it my duty to urge them very warmly to take up the study of analysis. A student who wishes nowadays to study geometry by dividing it sharply from analysis, without taking account of the progress which the latter has made and is making, that student, I say, no matter how great his genius, will never be a whole geometer.‡ He will not possess those powerful instruments of research which modern analysis puts into the hand of modern geometry. He will remain ignorant of many geometrical results which are to be found, perhaps implicitly, in the writings of the analysts. And not only will he be unable to use them in his own researches, but he will probably toil to discover them himself, and, as happens very often, he will publish them as new, when really he has only rediscovered them.

After the examples already given it seems useless to add more to justify these views. But there is one which I wish to cite, because it is instructive in the highest degree: *the geometry on an algebraic curve.*§ The idea of this geometry, *i. e.*, the properties of an algebraic curve which remain invariant under a rational transformation of the curve into itself, is first found

*[These words, as also the following, true in 1891, must now be modified. The present conditions in Italy are much better.]

† We must confess that they find the division already thus made in the catalogue of courses offered at the university. They find placed in oppposition *projective geometry* and *analytic geometry! higher analysis* and *higher geometry!*

‡ Remember that "geometer" in the wider sense of the term is synonymous with "mathematician."

§ [Cf. my Introduzione above referred to, and especially the very important report of Brill and Noether, "Die Entwickelung der Theorie der algebraischen Functionen in älterer und neuerer Zeit," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 3 (1893).]

in a work of analysis, in the great memoir of Riemann on the theory of abelian functions. It is here that we first find fully developed the notion of *genus* (it had already been introduced in substance by Abel in his capital memoir on his transcendents) and the demonstration that it is unchanged by such transformation. It is here that we find that representation of an algebraic function as the sum of abelian integrals of the second kind, which (thanks to Roch who completed the calculation of Riemann) gave to modern geometry one of its most important and fruitful theorems. It is here finally that we find so many noteworthy theorems, some of which when clothed in geometric form may well appear to be new to young geometers who have not studied thoroughly this profound and immortal work. New results in the geometry on a curve may also be found in the lectures of Weierstrass on abelian functions. And all through the geometry of algebraic curves the analytic researches on algebraic functions and their integrals have had applications of the greatest importance. It is sufficient to recall in this connection Abel's theorem (the natural bridge, as it has been called, between the algebraic and transcendental sides of this theory) and the inversion problem with which it is identified and the geometric applications of both of which Clebsch has exhibited.

Nor is the influence of Riemann on the applications of analysis to geometry confined to this. Indeed to his influence may be traced in many ways various new classes of transcendental functions, which are now defined and studied apart from the abelian functions. So Poincaré introduced certain uniform, analytic functions of one variable, which he called *fuchsian functions*, by means of which he is able to express the coördinates of the points of any algebraic curve. And this representation has already been used in studying curves;* it would however be desirable to see if more use can not be made of it. These functions are special cases of the more general class of *linearly automorphic* functions, as Klein calls them, *i. e.* of functions of one variable which remain unchanged when the variable is transformed by an infinite discontinuous group of linear transformations.† It is well worth while for young geometers to

*Cf. *e. g.* Humbert, "Application de la théorie des fonctions fuchsiennes à l'étude des courbes algébriques," *Journ. de mathém.*, 1886.

†Cf., besides the fundamental articles of Poincaré (in the first volumes of the *Acta Mathematica*), "Neue Beiträge zur Riemann'schen Functionentheorie," by Klein (*Math. Annalen*, vol. 21) [also Fricke-Klein, Vorlesungen über die Theorie der automorphen Functionen, Leipzig, 1897; Forsyth, Theory of Functions of a complex Variable, 2nd ed., Cambridge, 1900, last chapter.]

penetrate into the vast territory which these and other functions, both of one and of more variables, which are being introduced daily, open up for study; for it is certain that, even if they wish to confine their research to the field of geometry, the analytic knowledge thus acquired will be of the greatest service, and may perhaps give the key for the solution of most difficult geometrical problems.

V.

Here we may with propriety say a few words concerning the use of the *purely synthetic method* in geometrical research.

In any progressive development, either of a living organism, or of a social order, or of a science, two tendencies among others are always apparent: one to make ever finer distinctions in the structure and functions of the various parts, the other toward an ever greater connection and dependence among them.* It follows that the growth of the sciences, while on the one hand it increases their number by forming so many new sciences out of theories which before were parts of a single science, on the other hand continually increases the bonds between them and the means of mutual assistance which they can render.†

It is thus that among the many branches of mathematical progress in this century, aside from the new and instructive relations and applications established among them and in particular between analysis and geometry, was signalized the purification so to speak of these sciences, that is to say, the construction of an analysis based solely on the notion of number without admitting geometrical (or mechanical) considerations, and the construction of a projective geometry which makes no use of coördinates and avoids entirely the notion of number in the properties of position.

These *pure* methods are indeed of the greatest importance. It is beyond doubt that the mathematician can not be fully

* Or, as Herbert Spencer puts it in his general formula of evolution (First principles, 6th edition, p. 367), "a passage from an indefinite, incoherent homogeneity to a definite, coherent heterogeneity."

† Hence results what is perhaps the greatest difficulty which the student of today must meet, that of reconciling the division of labor, which is inevitable on account of the great development of the sciences, with the necessity of keeping up with the progress in several of them which are all intimately connected. Unfortunately their mental capacity and their physical strength make it possible for only a few to do this in an entirely satisfactory way. It is the young man who should use all his powers to approximate as near as possible to the scientific culture of which I speak; the studies pursued in youth are the ones which leave the deepest impression on the mind!

satisfied with the knowledge of a truth, unless he has succeeded in deriving it with the greatest simplicity and naturalness from the smallest possible number of known theorems, of independent postulates, avoiding every hypothesis and every means of demonstration which is not absolutely required for the purpose. By so doing we often gain besides the scientific advantage also a pedagogic advantage, in so far as a smaller amount of preliminary knowledge will be required of the student. Further we may note as a general principle that, when in our researches we adopt a single instrument, to the exclusion of every other, it will often happen that we make it more refined and more perfect, thus rendering it ever better adapted to new problems. And in this way the synthetic method, which was brought to perfection in the great works in which modern geometry was built up, has gained wonderful triumphs that show clearly the advantages which often result from its systematic use.

But in this line of thought the young geometer should be careful not to exaggerate. The heroic age, so to speak, of synthetic geometry, in which the object was not merely to give new results to the science, but in which all, from Poncelet to Steiner, from Chasles to Staudt, were engaged in the struggle to prove the usefulness of the geometric method to the analysts who were not willing to recognize it,—that age has passed ;* and nowadays battle is no longer necessary. That some one should be engaged in establishing synthetically results which others have already obtained analytically, is often very useful for reasons just hinted at ; but in doing this he should choose problems which present particular importance and noteworthy difficulties ; and then, these conditions being equal, he should always give the preference to investigations which are entirely new, and not become a mere translator of analytic writings into synthetic. And in his original research he should not feel obliged to cling exclusively to the geometric method, since ultimately the important things in the science are the results (unless the question is one of inventing new methods), and it

* And now seems to be the proper time to write its history ! — By the study of the great geometers some scholar should become inspired to search and relate how and by whose work have been obtained the most important advances in modern geometry. A detailed history of these advances would constitute a work of the highest interest. [This history, in part at least, has now been written. Cf. Ernst Kötter, "Die Entwickelung der synthetischen Geometrie von Monge bis auf Staudt," *Jahresber. d. Deutschen Math.-Vereinigung*, vol. 5 (1901).]

would be folly to avoid certain geometric questions merely because in the present state of the science they cannot be treated without the use of analytic instruments. Artificially to create difficulties, even though it may serve to sharpen the intellect and may therefore be of some use, is nevertheless not a plan to be followed in general by one whose chief object is the progress of the science.* For every piece of research one should choose freely the method which seems best adapted; often it will be found convenient to alternate between the synthetic method which appears more penetrating, more enlightening, and the analytic which in many cases is more powerful, more general or more rigorous; and it will frequently happen that the same subject will not be quite clear from all points of view unless treated by both methods. † Certainly here also the individual tendencies will make themselves felt and the same problem will seem to one more adapted to the synthetic treatment, to another to the analytic; thus Steiner and Cremona arrived at certain results synthetically at the same time that Sylvester and Clebsch obtained them analytically. But the most important thing is that the geometer should not be the slave of a single method and that he should put himself in a position where he can make use of any instrument to obtain an important result.

VI.

Just as purity of method takes a secondary place, when we are engaged merely with the discovery of a new result, so it often happens that in a first investigation we must sacrifice rigor (a sacrifice much more serious, since we are dealing with mathematics!).‡ Many times a scientific truth is

*For example, after Staudt has given a complete geometric theory of the imaginary elements, proving that all the fundamental properties of projective geometry hold for them, there is no reason in scientific research in this field (I am not speaking of those which are essentially pedagogical) for excluding imaginary elements, and still less for introducing them only in *couples* (of conjugate imaginary elements). It is understood that here, as always, I speak in most general terms, admitting the necessary exceptions.

† It is for this reason that in the present state of the science those treatises (or those courses) in which a subject is unfolded with a variety of methods, seem to me *in general* to be preferred to those in which a single method is adopted; thus algebraic curves, surfaces, etc., should, it seems to me, be presented to students analytically as well as synthetically, generally alternating the two methods according to the questions.

‡ We should not confuse lack of rigor in the methods with errors in reasoning or in the results. Errors are encountered only too often nowadays in

placed as it were on a lofty peak, and to reach it we have at our disposal at first only hard paths along perilous slopes whence it is easy to fall into abysses where dwells error; only after we have reached the peak by these paths is it possible to lay out safe roads which lead there without peril. Thus it has frequently happened that the first way of obtaining a result has not been quite satisfactory, and that only *afterwards* did the science succeed in completing the demonstration. Certainly also a mathematician can not be really content with a result which he has obtained by non-rigorous methods; he will not feel sure of it until he has rigorously proved it. But he will not reject summarily these imperfect methods in the case of difficult problems when he is unable to substitute better ones, since the history of the science precisely shows what service such methods have always rendered.

For example, the introduction of the imaginary and the infinitely distant elements into geometry proved to be most useful before it was justified with perfect logic. And Poncelet, who learned to employ to great advantage the principle of continuity for studying geometric figures, never succeeded in establishing it rigorously in spite of the many pages which he wrote on the subject.* Similarly in various, chiefly synthetic researches on curves and surfaces the writers have profited much by the considerations of infinitely near, or consecutive, points, of tangents, of r-ple points as the intersection of r branches, etc., which were far from being rigorous, at least in general and in the way in which they were presented. The same may be said of many processes of *enumerative geometry* † (in which we may

scientific work and are due generally to the haste with which the latter is produced. It is therefore well to advise young men not to show excessive haste in publishing their results! But the question of rigor, of which we speak above, is quite different.

*Cf. especially: "Considérations philosophiques et techniques sur le principe de continuité dans les lois géométriques" in vol. 2 of his Applications d'Analyse et de Géométrie; and the Traité des Propriétés projectives des Figures. From the introduction to this treatise I quote the following bit which applies to what we were saying above:

"N'est il pas, pour le moins, aussi nécessaire d'enseigner les ressources employées, à diverses époques, par les hommes de génie, pour parvenir à la vérité, que les efforts pénibles qu'ils ont été ensuite obligés de faire pour les démontrer selon le goût des esprits ou timides ou peu capables de se mettre à leur portée?

"Enfin, quel mal pourrait-il en résulter, surtout si l'on se montrait sévère à conclure, si l'on ne se payait jamais de demi-aperçus, si l'on n'admettait jamais *l'analogie et l'induction*, qui sont souvent trompeuses . . . ?"

†Cf. the very important work by Schubert: Kalkül der abzählenden Geometrie (Leipzig, 1879), in which these processes are collected and employed systematically.

include the last examples cited), such as the principle of counting constants and especially that of the invariance of number, in the application of which it is often assumed without a complete proof, that the number sought depends only on certain others, and therefore does not change (or become infinite) when they do not change; methods which have led able scientists, such as Jonquières,* Schubert and others, to splendid results, among which are many that the science would still be at a loss to reach by a more rigorous route.† Finally I will record that sometimes recourse has been had even to drawings and models of geometric forms in order to *see* certain properties (especially of form or of reality) which no one knew how to obtain by deductive reasoning alone.‡

Now in making use of similar means of investigation, the young investigator should be careful to remember that here, as in the difficult ascent of perilous heights, in order to avoid falling into an abyss or into error, he needs skill, prudence and practice. And having availed himself with the utmost caution of those means for his discoveries, he should seek to substitute for them rigorous proofs. But, as I have already said, I do not believe that he should refrain from engaging upon a line of research or from publishing a result, merely because he has been unable to proceed with absolute rigor. The greatest caution, I repeat, the largest number of verifications, etc., should be employed in order to avoid error; and also here the example of able mathematicians should help him to determine when he may have faith in a result which he has obtained; but then he can make

* Cf., for example, the determination of the number of groups of a linear series of point groups of a curve which have points of a given multiplicity; a determination contained substantially in the "Mémoire sur les contacts multiples des courbes de degré r, qui satisfont à des conditions données, avec une courbe fixe, etc.," by de Jonquières, *Crelle's Journal*, vol. 66 (1866).

† Speaking to young men, I like to cite among the more important results thus obtained, the one recently published by a young man, viz., the number of linear series of given index on a curve of given deficiency (cf. two notes by Castelnuovo in the *Rendiconti della R. Acc. dei Lincei*, summer, 1889). [Cf. the analytic justification of the geometric enumerative process of Castelnuovo in Klein, Riemann'sche Flächen II., p. 110 ff., lithogr. lectures, Göttingen 1892.]

‡ And there are other cases in which results relating to mathematics have been obtained by the use of physics. Even this is permissible; but it should be understood that the result thus obtained, though it be *scientific* (*physical*), is not therefore *mathematical*, will have only a relative and approximate value; however, it will prepare the way for the mathematical result, *i. e.*, for one established by a complete mathematical proof.

it known. Only it is necessary to unite *honesty* with caution, *i. e.*, to call particular attention to where the method used is open to question, so that no one may be induced to believe blindly in the result, but that it may be in itself an invitation to seek a more complete proof.*

VII.

At the outset of these remarks, following a quotation from Chasles, we spoke briefly concerning the extent and the fundamental importance which the notions of correspondence and algebraic transformations have in the geometry of algebraic spreads, and also of the facility with which by means of these we may obtain new forms and new propositions.

As regards their importance, we may consider this more particularly from three different points of view, corresponding to the following three functions of the correspondences themselves: 1° that of characterizing certain branches of geometry;† thus we have a projective geometry, in which the fundamental group of transformations is that consisting of the projectivities, a geometry of birational transformations, *i. e.*, the geometry of birational transformations in the plane or in space, etc., the geometry *on a curve* (which with the aid of function-theoretic considerations is made to correspond to the geometry of the conformal transformations of surfaces), the geometry *on a surface*, etc.; 2° that of transforming forms whose properties are known into others, of which we shall thus learn new properties; 3° that of generating new spreads by making use of algebraic correspondences among given forms, such as the loci of self-corresponding elements, or else the loci of the intersections (or the connections) of homologous elements, etc.

The second of the above divisions is suggested by the first, when in studying the properties of some geometric object from

* Among the things which must often be neglected in the first discovery of new theorems should be mentioned that of the exceptional cases. We frequently say that a proposition is true *in general*, without further specifying what is meant thereby, what are the *exceptions*. Of course this again lacks that exactness which is the greatest pride of mathematics. In the geometry of algebraic spreads a sufficiently precise meaning may be given to the phrase *in general*; we there mean that the exception takes place only in the case of those spreads whose coördinates (or parameters on which they depend algebraically) satisfy certain non-identical algebraic relations. If it is possible actually to exhibit these relations, that is, the conditions which characterize the exceptional cases, the mathematical ideal will be completely attained. But often we must content ourselves with merely the generic statement.

† *Cf.* Klein's Programm, already cited.

a certain point of view, *i. e.*, the properties which remain unchanged for a certain class of geometric transformations, we apply the latter to simplify the object of study, to make its study easier. This is the case, for instance, in the method of central projection as used in projective geometry; or when, in the study of the geometry on a curve, we make use of certain *normal* curves; or when the properties of certain linear systems of curves in a plane (in the geometry of the birational transformations of the plane) are derived from those of linear systems of lowest order. If, however, we compare the Traité des Propriétés projectives des Figures of Poncelet with the later treatises, for example with the Geometrie der Lage of Staudt,* we shall note how the importance of this method of demonstration has grown less in projective geometry, and how it has been deemed better to study directly the more general case, rather than to study first a particular and thence derive the general by projective transformations. And something analogous may already be seen in certain questions of the geometry of birational transformations.

This does not mean at all that the transformations lose in any way their importance. Far from it! It is merely the mode of application which changes. It is true, for example, that the study of *particular* rational curves and surfaces by means of their parametric representation, or their representation on a straight line or plane, and likewise the study of *particular* birational correspondences between two planes or two spaces, no longer commands the interest it had some twenty years ago. At that time they led to the consideration of new objects and new and very interesting problems, and the best geometers turned to them with profit. Nowadays to multiply the applications of the methods which were then invented is in general of secondary importance, unless the question is one of certain exceptional cases which present new difficulties. Given the linear system of plane curves representative of a surface, or given the homaloidal system of curves or surfaces which define a birational transformation of the plane or of space, it is in many cases nothing but a class-room exercise to derive therefrom the properties of the surface or of the birational transformation. And so in general, just as it is easy to imagine new loci and new geometric transformations, so also is it easy

* Translated into Italian by Pieri (Turin, Bocca, 1889).

to invent applications of correspondences to obtain new truths. We take a point A, join it to B, take the polar with regard to C, let it intersect with D, take the homologous point with regard to E, etc., and finally from A we obtain a point (or other element) A'; to the elements A of the first set will thus correspond the elements A' of a new set, and if A or A' is made to move in a certain way, A' or A will also move, etc. In this way a given figure or given property will be transformed into another, which will give rise to a third, and so on; and all without the least difficulty, mechanically as it were, with the regularity with which a pendulum swings.*

Now, though we remain in the field of algebraic correspondence, it is not with this sort of research that we should be engaged nowadays. There are indeed other problems of much greater importance!† The theory of the birational transformations of space is waiting for some one to attack certain general problems of prime importance, whose analogues for the plane have already been solved. It is waiting to be applied to the serious problem of the resolution of the higher singularities of twisted curves and surfaces, to the study of linear systems of surfaces, etc. The theory of the multiple correspondences between two planes or two spaces is still to be worked out. The study of algebraic correspondence between curves (distinct or coincident) is far from complete. And of that of the correspondences between two surfaces it may be said (in spite of a recent work by an able French mathematician) that it is still to be begun. In all of these fields may be found most vital questions to which the young geometer should address himself rather than to the exercises previously mentioned.

Even in the application of algebraic correspondences to the generation of geometric loci, important results have been found, and may still be found. But these correspondences must be chosen so as to lead to loci which interest by reason either of

* Hence one of my teachers (from whom I have received much of this advice which I am now giving to my students) used jokingly to call this kind of research *tic-tac-geometry*.

†[Several of the important questions which I have mentioned have since been discussed, in full or in part; the higher singularities of twisted curves and surfaces chiefly by myself and by B. Levi; algebraic correspondences between curves by A. Hurwitz; correspondences between surfaces by Picard, Painlevé, Castelnuovo-Enriques, Severi. Upon the geometry on an algebraic surface the not yet complete treatise of Picard and Simart, Théorie des Fonctions algébriques de deux Variables indépendantes (Paris, 1897, 1900, 1904) may be consulted with profit.]

their generality or of special circumstances. We may observe moreover that often the determination of the various characteristics (order, class, multiple points, etc.) of the forms thus defined does not present the least novelty or difficulty, because they are obtained by well known enumerative processes. The investigations therefore should not be limited merely to such a determination, but should include the examination of all the peculiarities of the loci obtained, with a view to solving the inverse question, *i. e.* of determining all the forms which may be generated in this way.

VIII.

Aside from the extension given to modern geometry by the use of transformations, we have already mentioned that derived from the enlargement of its field by the consideration of ever more extensive classes of spreads or systems. With this we may connect the consideration of the geometry of hyperspace, thanks to which the boundaries of geometry have been widened indefinitely. On this subject we may well say a few words,* on the one hand because during the past decade, especially after the appearance of the now famous memoir of Veronese,† several Italian geometers ‡ have turned to it with much energy, and chiefly the younger ones ; on the other hand because there are still, in spite of this, some even in Italy who are unable to accord to geometry of n dimensions its proper place. §

* [In my article "Mehrdimensionale Räume," which is to appear in the Encyklopädie der mathematischen Wissenschaften will be found more historical and bibliographical notes. Since the appearance of the present paper multi-dimensional geometry has spread more and more, so that now (among mathematicians!) its opponents have become rare, who at one time were so common. Not only the geometers, but also the abler analysts, no longer hesitate to make use of hyperspace in their researches.]

† "Behandlung der proj. Verhältnisse" etc., *Math. Annalen*, vol. 19, 1882.

‡ We could count at least twenty. [Now many more.]

§ To complete what I shall say in the following, and to support it with the voice of authority, I will repeat here the 4th Note (Über Mannigfaltigkeiten von beliebig vielen Dimensionen) of Klein's Programm, already referred to:

"Dass der Raum, als Ort für Punkte aufgefasst, nur drei Dimensionen hat, braucht vom mathematischen Standpunkte aus nicht discutirt zu werden ; ebensowenig kann man aber vom mathematischen Standpunkte aus jemanden hindern, zu behaupten, der Raum habe eigentlich vier, oder unbegränzt viele Dimensionen, wir seien aber nur im Stande drei wahrzunehmen. Die Theorie der mehrfach ausgedehnten Mannigfaltigkeiten, wie sie je länger je mehr in den Vordergrund neuerer mathematischer Forschung tritt, ist, ihrem Wesen nach, von einer solchen Behauptung vollkommen

We may distinguish three ways in which hyperspace has presented itself to geometers; and to these correspond as many ways of defining the points of a linear space of n dimensions. In the first place, on the basis of the method of coördinates, we regard the points of a line, of a plane, or of space, as represented by analytic spreads composed of all possible values of one number, of two numbers, or of three, so that systems of equations in one, two, or three variables are represented by certain sets of points, etc. We are led naturally to extend the language of geometry to the case of any number n of variables, still using the word *point* to designate any system of values of n variables (the *coördinates* of the point), the word *space* (of n dimensions) to designate the totality of all of these points or systems of values, *curve* or *surface* to designate the spread composed of points whose coördinates are given functions (with the proper restrictions) of one or two parameters (*straight line* or *plane*, when they are linear fractional functions with the same denominator), etc. Such an extension has come to be a necessity in a large number of investigations, * in order as well to give them

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unabhängig. Es hat sich in ihr aber eine Redeweise eingebürgert, die allerdings dieser Vorstellung entflossen ist. Man spricht, statt von den Individuen einer Mannigfaltigkeit, von den Punkten eines höheren Raumes, etc. An und für sich hat diese Redeweise manches Gute, insofern sie durch Erinnern an die geometrischen Anschauungen das Verständniss erleichtert. Sie hat aber die nachteilige Folge gehabt, dass in ausgedehnten Kreisen die Untersuchungen über Mannigfaltigkeiten mit beliebig vielen Dimensionen als solidarisch erachtet werden mit der erwähnten Vorstellung von der Beschaffenheit des Raumes. Nichts ist grundloser als diese Auffassung. Die betr. mathematischen Untersuchungen würden allerdings sofort geometrische Verwendung finden, wenn die Vorstellung richtig wäre, — aber ihr Wert und ihre Absicht ruht, gänzlich unabhängig von dieser Vorstellung, in ihrem eigenen mathematischen Inhalte.

Etwas ganz anders ist es, wenn Plücker gelehrt hat, den wirklichen Raum als eine Mannigfaltigkeit von beliebig vielen Dimensionen aufzufassen, indem man als Element des Raumes ein von beliebig vielen Parametern abhängendes Gebilde (Curve, Fläche, etc.) einführt.

Die Vorstellungsweise, welche das Element der beliebig ausgedehnten Mannigfaltigkeit als ein Analogon zum Punkte des Raumes betrachtet, ist wohl zuerst von Grassmann in seiner Ausdehnungslehre (1844) entwickelt worden. Bei ihm ist der Gedanke völlig frei von der erwähnten Vorstellung von der Natur des Raumes; letztere geht auf gelegentliche Bemerkungen von Gauss zurück und wurde durch Riemann's Untersuchungen über mehrfach ausgedehnte Mannigfaltigkeiten, in welche sie mit eingeflochten ist, in weiteren Kreisen bekannt.

Beide Auffassungsweisen — die Grassmann'sche wie die Plücker'sche — haben ihre eigenthümlichen Vorzüge; man verwendet sie beide, zwischen ihnen abwechselnd, mit Vortheil."

* I will cite as examples only the investigations concerning the systems of algebraic equations (algebraic spreads in any space) of Salmon (Geometry of

the greatest generality, as to preserve in them the intuitive character of geometry. But it has been noted that in such use of geometric language we are no longer constructing truly a geometry, for the forms that we have been considering are essentially analytic; and that, for example, the general projective geometry constructed in this way is in substance nothing more than the algebra of linear transformations.*

A geometric method of arriving at the notion of hyperspace, following the ideas of Plücker, is to consider as elements (*points*) of a spread (of a *space*) geometric forms of ordinary space, such as groups of points, curves, surfaces, ··· which depend on any arbitrary number of parameters. It is thus that the straight lines of ordinary space may be thought of, according to Plücker, as *points* in a space of four dimensions. However, if in this representation it be desired to avoid exceptional elements, *i. e.*, if it be desired to represent *linearly* a manifold of ∞^n ordinary geometric forms by the points of a (linear) hyperspace, it is necessary that the manifold be linear. Thus an involution of points on a straight line of the nth species and any order (for example that composed of all the groups of n points), a linear system of ∞^n plane curves, or surfaces, or connexes, etc., is from this point of view a (linear) space of n dimensions, such that the spreads therein contained are nothing but partial spreads of the involution, or of the linear system in question. This means of representation offered itself spontaneously to the geometers who wished to go more deeply into the questions concerning infinite systems of plane curves (or surfaces).†

Three Dimensions and Modern Higher Algebra), the work of Noether, "Zur Theorie des eindeutigen Entsprechens algebraischer Gebilde von beliebig vielen Dimensionen," *Math. Annalen*, vol. 2 (1869), and the memoir of Clebsch, "Ueber eine Fundamentalaufgabe der Invariantentheorie," *Gött. Abhandlungen*, vol. 17 (1872)), referring for other instances to the historical monograph of Loria.

*This is a distinction, not a reproach. It is *mathematics* that is being made.

†Thus Cayley makes use of it in the first paragraph of his memoir "On the curves which satisfy given conditions," *Phil. Transactions*, vol. 158, 1867 [Coll. Math. Papers, vol. VI, p. 191], and he returns to the subject in "A memoir on abstract geometry" (ibid., vol. 160 (1869) [ibid. p. 456]), where speaking of the importance which *abstract geometry*, *i. e.*, of n dimensions, should have, he says: "The science presents itself in two ways, as a legitimate extension of the ordinary two- and three-dimensional geometries; and as a need in these geometries and in analysis generally." The same idea of the application of hyperspace to infinite systems of plane curves (or of surfaces) is explained clearly with some examples by Halphen at the end of his "Recherches de géométrie à n dimensions" (*Bulletin Soc. Math. de France*,

It is clear that, in this way of thinking, the geometry of hyperspaces no longer presents any novelty of idea; it enters into ordinary geometry, in the treatment of the spreads of forms which occur therein.

Finally we may regard space of n dimensions as defined in the same way as ordinary space, except that we omit the postulate concerning the three dimensions, and consequently modify some of those referring to the straight line and plane. Then the points of the hyperspace are points of the same nature as those which we think of in ordinary space, and are no longer purely analytic forms or geometric forms of various kinds.* To this point of view are committed in particular those who have discussed the foundations of geometry by determining what postulates of the ordinary science can be dispensed with without too much inconvenience. And it is just here that a confusion has frequently made itself apparent with the (physical or philosophical, but not mathematical) question concerning the number of dimensions which ordinary (physical) space *actually* possesses (see the words of Klein previously quoted in a footnote).†

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vol. 2 (1873)); it may further be found in the memoir by Clifford, "On the classification of loci," *Phil. Trans.*, vol. 169 (1878) [Math. Papers, p. 305] (in which the *points* of a hyperspace are defined and interpreted in the applications as elements of any nature whatever, curves, surfaces, complexes, etc); and in several other recent works, as for instance in that of Study, "Ueber die Geometrie der Kegelschnitte, insbesondere deren Charakteristikenproblem," *Math. Annalen*, vol. 27 (1886).

*Cf. Veronese, *loc. cit.*, and "La superficie omaloide normale," etc. (*Memorie Lincei*, 3d series, vol. 19), the note at the bottom of the 2d and 3d pages. The fundamental hypothesis (which is there laid down) for this method of procedure, *i. e.*, that outside of the straight line, of the plane, of ordinary space, ···, there are always points (which, connected with spaces not passing through them, give rise to higher spaces) is allowable in mathematics, because it does not contradict any of the preceding postulates (after the above mentioned modifications have been made in the ordinary form). [Cf. also Veronese, Fondamenti di Geometria a più Dimensioni, (Padova, 1891), or the German translation (Leipzig, 1894). The same notion of hyperspace is the basis of Schoute, Mehrdimensionale Geometrie (Leipzig, 1902).]

†[In regard to the *foundations of geometry*, the books by Pasch and by Peano, and since the publication of this article the book by Veronese and the papers of Pieri, Hilbert and others have led mathematicians in recent years more and more to consider geometry from an *abstract*, purely *logical* or *deductive*, point of view, detaching it entirely from every physical consideration. With this tendency, the *primitive* or *undefined* concepts, such as for example *point* or *movement*, come not at all to correspond in any way to determinate physical entities; they may receive instead many different interpretations. Thus in the *primitive* propositions or *postulates* also there exists an analogous liberty. Recourse may still be had to *spatial intuition*

Every one of these ways of considering the linear space in geometry has some advantages; and in particular it may be noted that the first is very general and very simple, but analytical, whereas the third is geometrical and entirely intuitive; and the second is the one which lends itself more readily to numerous applications in ordinary space. But this distinction of points of view does not imply different treatments of (projective) geometry of n dimensions. Indeed for the mathematician this distinction has no real importance. He may well avoid it entirely, and work in hyperspace without making a fixed choice between the three definitions; and he may with perfect right retain all of them in order to have a greater variety of representation and interpretation of his results.

IX.

From what has been said it is evident that the geometry of n dimensions has no mathematical characteristics essentially different from those of ordinary geometry. Spaces of 4, 5, $\cdots$ dimensions, as we have defined them, exist for the mathematician precisely in the same way as space of 3 dimensions exists; and they may be studied by the same methods. Thus an algebraic spread M_k (of order m) in an S_n is an analytic or geometric form existing in the same sense as exist the n algebraic functions of k independent parameters by means of which it is defined, or even as there exists an ∞^k algebraic system (of index m) of plane curves or surfaces within an ∞^n linear system, etc.

It would moreover be absurd to say that the figures of hyperspace are of less importance (mathematically) than those of ordinary space. Can any one claim that plane geometry is more important than solid, or that the theory of functions of one or two variables is more important than that of any number n of variables, or that the study of an ∞^3 of forms is of more value than that of a manifold of any order of infinity ∞^n?

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for guidance or as a means of research; not so however for the demonstrations; these must be exclusively *deductions* by *pure logic*.

Following this method the points of a space of 4, or 5, $\cdots$ dimensions are treated as above stated in the same way as those of S_3, the system of postulates being slightly modified. The problem which I have proposed to students in this part of my article: "to lay down a system of independent postulates which will completely define linear space of n dimensions, so that therefrom may be deduced the representation of the points of this space by means of coördinates" was immediately discussed by two of those who were attending my lectures: Amodeo and Fano. Cf. also Killing, Einführung in die Grundlagen der Geometrie, Paderborn, 1893 and 1898.]

Naturally however we shall not say that an investigation, *merely because it has reference to hyperspace*, is more important * than an investigation relating to ordinary space! The comparison is possible only when the first investigation is reduced to the second, taking $n = 3$; and then there is no doubt that to greater generality will correspond greater importance. On the other hand if we compare the *whole* content of geometry of n dimensions with the *whole* of ordinary geometry, we may say the latter is contained as a particular case in the former; but we should add that all the propositions of the former must be contained in substance in the ordinary geometry, as properties relating to ∞^n systems of ordinary geometric forms.

Now, to the remarks on various tendencies in research which we made first with reference only to ordinary geometry, but which we must think of as extended immediately to geometry of n dimensions, it is proper to add a distinction which naturally arises between the investigations to which the introduction of hyperspace has led. Of the infinite series of new forms and new problems for study which this has brought to light, we find that some are derived by analogy and generalization from those of ordinary space, whereas others arise from concepts which can not be found in the latter. And thus we encounter investigations which are relatively easy to carry on because in them the analogy to ordinary space serves as a guide to the generalization of the methods and results; but we meet also problems which contain really new difficulties. In order that the edifice (whose construction is only just begun) may be finished, we must undertake both classes of problems.

Consequently it is a mistake to hold that geometry of hyperspace is nothing more than an easily made extension of ordinary geometry; just as if the whole geometry of ordinary space were a simple and easy generalization of plane geometry! And those geometers are in the wrong who (to avoid difficulties!) expressly confine themselves to the first class of investigations, and by so doing create the above impression among their readers. It is indeed true that, thanks to hyperspace, those who love easy reasonings have seen fit to increase their number in an extraordinary way. How many new generalizations have they not seen fit to make, so easy that they could

* Nor more difficult! Experience shows that especially for young students it is fairly easy to become accustomed to the constructions and processes of reasoning relating to geometry of n dimensions.

be announced offhand! What new special forms have they not constructed! What particular transformations, what projections to space of lower * dimensions could they not make them undergo! But here again I must repeat that, though we must not without further consideration condemn a body of reasoning merely because it is easy, nevertheless we must not allow ourselves to be lured on merely by easiness; and we should take care that every problem which we choose for attack, whether it be easy or difficult, shall have a useful purpose, that it shall contribute in some measure to the up-building of the great edifice; that it shall not be one of those simple exercises, those useless generalizations, which are so easy to imagine and to carry out, and which, as they can never become an essential part of the science, so never deserve any sincere praise!

X.

Unlimited space, without any restriction on the number of dimensions, is the medium in which nowadays geometric forms should be considered. By doing this, we obtain besides the great advantage of generality, the further advantage of removing the obstacles which until recently prevented a geometer, aware of the utility for plane geometry of certain stereometric considerations, from seeking in higher space analogous applications for use in ordinary space.† In hyperspace there are no limits; every space is contained in a higher one; and in the latter we may seek for forms which will simplify the study of given forms in the former, by producing them as projections, or as sections, or as apparent outlines, etc. It is here that we find one of the important applications of the geometry of hyperspace to that of ordinary space;‡ but not here alone. We may see at once from the second defini-

* And often neglecting the *inverse problem*, *i. e.*, to see when the form of lower space may be truly regarded as the projection of the form of the higher space!

† Moreover (I may well repeat it) the scientific importance of the geometry of hyperspace would still exist perfectly even without these applications.

‡ It seems that the first instance of such an application is due to Cayley, who in the note: "Sur quelques théorèmes de la géométrie de position," *Crelle's Journal*, vol. 31 (1845), p. 218 [Coll. Math. Papers, vol. 1, p. 321], observes that certain configurations of ordinary space or of the plane may be obtained as sections of the configuration determined by any number of points in any hyperspace. Other applications were then noticed by other mathematicians; but the work which by the use of projection from higher spaces into ordinary space has truly created an epoch, is that of Veronese.

tion given above of higher spaces, that all results relating to these may be translated in numerous ways into ordinary geometric propositions, according to what geometric forms of ordinary space may be taken to replace the points of the hyperspace.* Besides, it is often necessary to turn to higher spaces to find the most convenient representation of certain kinds of manifolds. Thus the study of a linear system of curves in a plane or of surfaces of ordinary space (from the standpoint of the birational transformations) finds a convenient image in the (projective) study of the hyperspatial surface or spread which is represented thereby. For the

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Against this method the following objection has been raised. If a form of ordinary space is studied by regarding it as the projection of one belonging to a higher space, we thus introduce into our reasoning a new postulate — the existence of points outside of ordinary space — and therefore the results obtained will not have quite the same value that they would have if they had been obtained by deductions made without leaving ordinary space and the postulates relating thereto. This objection which appears valid if the hyperspaces have been defined by the third method given above, is on the contrary removed at once if recourse be had to the other two. Let us consider a representation of the ordinary points on an ∞^3 spread consisting either of sets of values of three numbers, or of conveniently chosen ordinary geometric forms. This can be done *without any new postulates*. Then let us consider (and this again is possible without adding any new postulates) this ∞^3 spread as being contained in an ∞^n spread (analytic or geometric), and to the latter let us refer our hyperspace reasonings, from which we draw conclusions concerning the ∞^3 spread. Finally we will return to ordinary space by means of the representation of the latter on the ∞^3 spread.

Thus, if with the three coördinates x_1, x_2, x_3 of an ordinary point we associate $n-3$ zeros, we may represent this very point as an *analytic* point $(x_1, x_2, x_3, 0, \cdots 0,)$ of an analytic space of n dimensions, whose points have the coördinates $(x_1, x_2, x_3, \cdots, x_n)$.

Or again, to every point x of ordinary space considered as an envelope of planes ξ whose equations are $\xi_x = 0$, let us join a fixed envelope of planes of class $m-1$, $\phi(\xi) = 0$. We shall obtain an envelope of class m, $\xi_x \cdot \phi(\xi) = 0$, which, by changing the point x, will describe a linear system of ∞^3 such envelopes. By taking m sufficiently large, this system may always be regarded as contained in an ∞^n linear system and then we may take this as the representation of the hyperspace S_n used in our reasoning. The projections and sections, which for simplicity and brevity of statement are made to refer to the points of an S_n, could as well be translated immediately into constructions relating to the envelopes of this linear system, and would thus lead to the same results in ordinary space.

*That geometer is surely not familiar with the variety of interpretations of which the geometry of hyperspace is capable, who recently fell into error in regard to some of the simplest problems concerning ∞^1 systems of plane curves or surfaces of index m, whose solution is contained implicitly in well-known theorems on curves of order m in hyperspace; or who regarded as new the determination of the ruled surfaces of given degree and maximum deficiency, whereas this may be considered as contained in the determination of curves (viz., the M_4^2 of straight lines) of given order and maximum deficiency.

geometry on an algebraic curve of genus p, and particularly for the study of the linear series of point groups on this curve, it is of advantage to turn to certain curves of the various spaces, which represent these linear series, and particularly to the normal curve of order $2p - 2$ in the space S_{p-1}.* Analogous remarks may be made concerning the geometry on a surface, and more generally concerning the geometry on any algebraic spread.† Moreover in these the use of hyperspace will be necessary to represent in a real field the distribution of the points, real and imaginary, of the surface (or spread), *i. e.*, of the values of an algebraic function of *two* (or more) variables, in the same way that in the geometry on a curve, *i. e.*, in case of algebraic functions of *one* variable, recourse has been had to Riemann surfaces. The investigation of the *analysis situs* in hyperspace will therefore find applications of the greatest importance in the geometry on algebraic spreads, *i. e.*, in the theory of algebraic functions of two or more variables.

In this way, making use on the one hand of the methods of transformation of modern geometry, of the considerations of hyperspace, etc., and on the other hand of the concepts due to analysis, the forms which we consider will present themselves under such a variety of aspects and points of view, and will be capable of transformation in so many ways that, while the instruments of study grow ever more numerous, at the same time the interest and importance of the investigations become far greater and acquire besides a singular elegance and charm. Thus the algebraic ∞^1 form (*algebraisches Gebilde*) of genus p is represented by an algebraic curve in any space, or by a ruled surface, or by a ∞^1 system of curves or surfaces, etc.; or by a class of algebraic functions or of abelian functions; or by a system of automorphic functions; or by a Riemann

* In this connection I would like to recall that in the recent Vorlesungen über die Theorie der elliptischen Modulfunctionen by Klein (and Fricke), (vol. 1, Leipzig, 1890; III. Abschnitt) the geometric representation of the algebraic ∞^1 form and of the theory of the algebraic functions and their integrals, and thence of the linear series of point groups, etc., is given precisely by means of curves of all spaces (instead of merely the plane curve, as was customary for a long time). [This example has since been followed in other analytic treatises.]

† I have frequently made mention of the geometry on an algebraic spread, curve, surface, etc., because I wish to call the attention of students to it; it is indeed a subject of prime importance for the whole of mathematics — for analysis as well as geometry — and it would be well if many directed their energy to it!

surface, or by a sphere with p handles, or with p holes, or by a plane curvilinear polygon generating a fuchsian group,* etc., and all these things are equivalent; by means of geometric or analytic transformation, by continuous deformation, by conformal representation, they change one into the other; and from all these representations the study of the algebraic form may derive help.

Those students who, following these suggestions, succeed in obtaining possession of the most varied instruments of research and in comprehending the science from all possible points of view, will attain therefrom the most brilliant success.

TURIN, *February*, 1891.

REPLY TO PROFESSOR SNYDER'S REVIEW OF STUDY'S GEOMETRIE DER DYNAMEN.

IN the January number of the BULLETIN Professor V. Snyder, of Cornell University, lavishes praise and censure on the author of the book under his review, using vigorous terms in both respects. I beg leave to raise, in the present instance, some objections to this mode of criticism.

Preceding reviewers have successfully tried to convey to their readers, space permitting, an idea of the actual contents of the book in question.† Dr. Snyder takes a way of his own. He confines himself mainly to a consideration of the fundamental conceptions of the author's line geometry, as developed in the second half of the volume, and to an enumeration of subjects treated therein. While the latter is very incomplete, omitting, *e. g.*, the most important part of the book (the appendix, referring to kinematics), the former reproduces, not so much the contents of the book itself as those of a paper by Professor E. Müller, and besides develops certain ideas dealing with non-euclidean geometry. By these means Professor Snyder endeavors to make the author's treatment "easier to understand."

* In particular for the elliptic form ($p=1$), the annular surface, the parallelogram, the (2, 2) correspondence between two variables, etc.

† Such reviews have been published by Professor Zindler, of Innsbruck, (*Monatshefte für Mathematik und Physik*, 1903—5 pp.), Professor Wirtinger, of Vienna, (*Zeitschrift für Mathematik und Physik*, vol. 49 (1903)—4 pp.), Professor Daniele, of Pavia, (*Bolletino di bibliografia ecc.*, Dic. 1903—14 pp.).

What help the secondary source quoted may offer, will be inferred from the fact that Dr. Müller, when he wrote his account, had before him only the first and smaller half of the volume, containing very little of the subject in question.

The reviewer's own development is introduced by the remark that "no hint is given the reader" by the author himself "regarding the origin of the new idea," and it concludes with the repeated assertion that "the development here traced is not mentioned by Professor Study" and that "perhaps he made no use of it." It is, however, pointed out in the prefaee that ideas of non-euclidean geometry were of considerable value to the writer — a "hint" which is, by the way, repeated in *many* places scattered through the book (see article "analogies" in the index) — and there is, moreover, a reference to pages 440 and 441, where the author confesses his indebtedness to other writers. In this place, to which the reviewer would also have been led if he had looked up the item "Non-Euclidean Geometry" in the index, he would have found immediately *the very thing he is missing*, with the exception of some errors, for which he alone is responsible. In the same place he would also have found mention of a paper by the author, bearing the date of 1900, in which the topic is treated more fully.

The haste in which this review has been executed is further shown by Dr. Snyder's statement that the author's aplanar chain congruence is "*dual to*" the congruence of axes in a sheaf (bundle) of linear complexes. As a matter of fact the aplanar chain congruence is *identical with* the congruence of axes in a bundle of complexes (fulfilling certain inequalities) and is even defined as such, on page 323 (cited in the index). This curious mistake causes the astonishing statement that the aplanar chain congruence and its limiting cases are "fully discussed" while the congruence of axes receives, comparatively, "too little attention"! To appreciate this confusion rightly, it is to be noticed that the congruence whose very definition is distorted by the critic, is one of the fundamental conceptions of the theory in question (treated in more than eighty pages, and dealt with at length also, under the same mode of generation, in the previous literature cited by the reviewer). But behold one more example!

For certain purposes the author thought it appropriate to count each straight line twice and accordingly cover the line space with two "sheets" (pages 224, 231–233). This simple

idea, which is, of course, totally different from the so-called orientation of a straight line (*i. e.*, the discrimination of its two directions), Professor Snyder explains as follows: He first considers a (real) line, in the finite domain, as being given by two points λ_1, λ_2 on a sphere. Then "*in order to include all the lines of space*" he makes the radius of the sphere "infinitely large" and goes on to say: "*In order to distinguish between the two points* λ_1, λ_2 *on the surface of the sphere and insure that the line connecting them does not lie entirely at infinity,* it is convenient to think of all the lines of space being double, or arranged in two sheets." (The italics are the present writer's.)

These examples will suffice to give the reader an idea of the oversights and errors that are numerous in this review.

Professor Snyder has also seen fit to make comments on the author's style, including one of a personal nature. That the text is hard and the reasoning sometimes complicated the writer does not seek to deny.* He cannot help regretting, however, that the reviewer did not find it worth his while to overcome even the lesser difficulties before publishing his review. Under these circumstances the words of praise which are distributed with liberal hand will have little weight.

E. STUDY.

BONN, *April*, 1904.

My statement that the congruence (3, 2) is dual to the congruence of axes in a linear sheaf is erroneous; this fact however does not affect the validity of the subsequent part of my review. My definition of the congruence (3, 2) and the inter-

*The chief hindrance to the reader lies probably in the terminology. It has already been conceded in the preface that the terminology is actually somewhat too extensive (a fact not mentioned by the reviewer). Dr. Snyder's statement, however, that the number of "new" terms in the first part of the volume alone exceeds one hundred, may be literally correct and yet fail to do justice to the writer. Some expressions are to be kept in memory for a few pages only, others are obvious derivatives from preceding terms, and others again are as a matter of course transferred from one system of constructions to the next. To take an example: It would be hardly fair to say that *nine* new terms are contained in the words: Keil, Quirl, Linien-Kreuz; eigentlicher Keil, eigentlicher Quirl, eigentliches Linien-Kreuz; uneigentlicher Keil, uneigentlicher Quirl, uneigentliches Linien-Kreuz. Some of my colleagues, who kindly did the counting for me, are of opinion that with a reasonable method of counting the "new" terms in the first part (covering 115 pp.) may be estimated to amount to 40 or 50 — a number still too large, no doubt.

pretation of the symbols employed are taken literally from Professor Study's book, page 461.

The transition from G_{16} to G_{17} was discussed by Professor Klein in the lecture of June 15, 1903, of his course on the encyclopedia of geometry. My own explanation would have been improved had I further quoted from this lecture.

In regard to Professor Müller's paper I will add that the "first and smaller half" of Professor Study's volume contains pages 1–240. The entire development of ε and of G_{17} is contained in pages 225–240.

I regret having used the word self-conscious. While I desired to emphasize that the book was hard to read on account of the style, I had neither cause nor inclination to make any personal reflection on its author.

VIRGIL SNYDER.

NOTES.

THE sessions of the mathematical department of the international congress of science at St. Louis, will open on Tuesday, September 20, with addresses by Professors MAXIME BÔCHER and JAMES PIERPONT. Professor Bôcher's address will be of a theoretical and critical character; Professor Pierpont will present a historical resumé of mathematical progress in the nineteenth century. On the following days of the meeting (Wednesday–Friday) the three sections of the department will meet separately, but at different hours. The principal speakers before the sections will be: Section 1, analysis and algebra, Professors E. PICARD and H. MASCHKE; Section 2, geometry, Professor G. DARBOUX and Dr. EDWARD KASNER; Section 3, applied mathematics, Professors H. POINCARÉ and L. BOLTZMANN. A detailed announcement of titles of the several addresses, arrangement of hours, etc., will be made in June by the exposition committee on congresses, of which Professor SIMON NEWCOMB is president.

THE National academy of sciences held its annual meeting at Washington, D. C., April 19–23. The only mathematical paper was by Mr. C. S. PEIRCE, "On the simplest branches of mathematics." The officers of the preceding year were re-

elected. Professor W. F. OSGOOD, of Harvard University, was one of the new members elected at this meeting. Professor G. H. DARWIN, of Cambridge University, was elected foreign associate of the academy.

THE British association for the advancement of science will meet in Cambridge, England, August 17, 1904. Professor H. LAMB is president of section A, mathematics and physics.

THE officers of the French mathematical society for the present year are: President, Professor E. CARVALLO; secretaries, Professors R. BRICARD and A. GRÉVY. The new annual list of members of the society contains 304 names.

AT the Breslau meeting of the German mathematical society, September 19–24, the president, Professor H. WEBER, will be represented by Professor F. KLEIN and the secretary, Professor A. KRAZER, by Professor A. GUTZMER.

AT the meeting of the London mathematical society held on April 14, the following papers were read: By Professor F. MORLEY, "On a plane quintic curve"; by Mr. H. T. HAVELOCK, "Mathematical analysis of wave propagation in isotropic space of p dimensions"; by Rev. E. W. BARNES, "On functions generated by linear differential equations of the first order"; by Mr. G. W. HARDY, "Note in addition to a former paper on conditionally convergent multiple series"; by Mr. H. HILTON, "Spherical curves II"; by Mr. P. W. WOOD, "Perpetuant syzygies of degree four"; by Rev. F. H. JACKSON, "Transformations of the function $F(\alpha, \beta, \gamma, x)$"; by Professor G. A. MILLER, "An extension of Sylow's theorem"; by Rev. J. CULLEN, "Note on a system of linear congruences"; by Rev. F. H. JACKSON, "An extension of Neumann's addition theorem for Bessel functions"; by Dr. H. F. BAKER, Transvectant operators in connection with binary forms"; by Lieut. Colonel A. CUNNINGHAM, "Factors of $13^{39}-1$."

PROFESSOR OSKAR BOLZA'S Lectures on the Calculus of Variations are in press and will appear in July. The book, which will contain from 250 to 300 pages, is based upon the lectures delivered by Professor Bolza at the Ithaca colloquium in 1901. It treats that class of problems of the calculus of variations in which definite integrals are considered, which depend upon a plane curve and contain no higher derivatives than

the first. The following subjects are discussed in detail: Euler's differential equation, Legendre's condition, Jacobi's condition, sufficient conditions; the case of variable end points; Weierstrass's theory of the problem in parameter representation; Kneser's theory; isoperimetric problems; Hilbert's existence theorem. The Lectures will be published as one of the volumes of the Decennial Publications of the University of Chicago.

UNDER the title of Rational Geometry, Professor G. B. HALSTED, of Kenyon College, Ohio, has prepared a school geometry based upon the researches of Hilbert. The book includes both plane and solid geometry and contains about two hundred and seventy pages. It will be published by John Wiley and Sons in a few weeks.

A THREAD model of the quintic scroll having three double conics, with the simple conic marked by a series of small beads on the generators, has been constructed by Mr. R. P. BAKER, of Union Academy, Anna, Ill. The model is about twelve inches cube, and brings out the form of the surface very clearly. Duplicates can be procured from Mr. Baker.

DR. THOMAS MUIR has recently issued a reprint of his paper entitled "A third list of writings on determinants," published in the Report of the South African association for the advancement of science, for 1903. Dr. Muir's first two lists are well known, having appeared in the *Quarterly Journal of Mathematics*, volumes 18 and 19. The first list covered the period 1693–1880 and contained 589 titles; the material collected was used in the author's work on the history of the theory. The second list covered the period 1784–1885, supplying 84 titles not given in the first list, and adding 176 for the five years 1880 to 1885. The present list covers the period 1748–1900, adding 138 titles to the first list, 56 to the second, and 711 titles since 1885. The total list now includes 1,744 titles, and is thought to be complete to the close of the nineteenth century.

THE universities below offer during the academic year 1904–1905 advanced courses in mathematics as follows:

BROWN UNIVERSITY.—By Professor H. P. MANNING: Theory of functions, three hours; Differential equations, three hours; Advanced algebra, three hours.

BRYN MAWR COLLEGE.—By Professor CHARLOTTE A. SCOTT: Modern pure geometry, two hours; Theory of transformation and geometry on a curve, two hours.—By Mr. J. E. WRIGHT: Differential equations, two hours; Theory of continuous groups, two hours.—By Dr. ISABEL MADDISON: Analytic geometry of three dimensions, one hour.

UNIVERSITY OF CALIFORNIA.—By Professor I. STRINGHAM: Theory of functions of a complex variable, three hours; Logic of mathematics, three hours; Seminar, two hours.—By Professor G. C. EDWARDS: Differential equations, three hours.—By Professor M. W. HASKELL: Elliptic and other special functions (first half year), three hours; Analytic projective geometry, three hours; Quaternions (second half year), three hours.—By Professor C. A. NOBLE: Theory of equations, two hours; Advanced analytic geometry (first half year), three hours; Differential geometry, three hours.—By Mr. A. W. WHITNEY: Theory of probabilities, one hour.—By Dr. D. N. LEHMER: Synthetic geometry (first half year), three hours; Theory of numbers, three hours.—By Dr. T. M. PUTNAM: Partial differential equations (first half year), two hours; Theory of groups (second half year), three hours.

UNIVERSITY OF CHICAGO. — Professor O. BOLZA will lecture on calculus of variations during the summer session, instead of integral calculus as announced in the May BULLETIN.

CLARK UNIVERSITY. — By Professor W. E. STORY: Advanced analytic geometry, five hours; Algebraic invariants (first half year), two hours; Finite differences (second half year), two hours; Seminar. — By Professor H. TABER: Theory of functions and elliptic functions, five hours; Transformation groups (second half year), two hours; Seminar. — By Mr. J. DE PEROTT: Theory of numbers (first half year), two hours; Algebraic substitutions and their application to the theory of equations (second half year), two hours.

COLUMBIA UNIVERSITY.—By Professor T. S. FISKE: Advanced calculus, three hours; Functions defined by linear differential equations, three hours.—By Professor F. N. COLE: Introduction to the theory of functions, three hours; Theory of invariants, three hours.—By Professor R. S. WOODWARD: Advanced theoretical mechanics, two hours; Theory of the po-

tential function, two hours; Mathematical theory of elasticity, two hours.—By Professor D. E. SMITH: History of mathematics, two hours.—By Professor JAMES MACLAY: Application of the calculus to the theory of surfaces, three hours.—By Professor C. J. KEYSER: General theory of assemblages, three hours.—By Dr. G. H. LING: Theory of numbers, probabilities, three hours.—By Dr. EDWARD KASNER: Differential equations and continuous groups, two hours.

CORNELL UNIVERSITY.—By Professor L. A. WAIT: Advanced analytic geometry, three hours; Advanced differential calculus, three hours.—By Professor J. W. JONES: Algebra, three hours.—By Professor J. MCMAHON: Theory of sound, three hours; Theory of potential and spherical harmonics, two hours.—By Professor J. H. TANNER: Theory of equations, two hours.—By Professor J. I. HUTCHINSON: Projective geometry, three hours; Theory of functions, three hours.—By Professor V. SNYDER: Algebraic plane curves, three hours.—By Dr. W. B. FITE: Theory of groups, three hours; Definite integrals, two hours.—By Dr. C. N. HASKINS: Advanced differential equations, three hours; Calculus of variations, two hours. The Oliver mathematical club will hold weekly sessions.

UNIVERSITY OF MINNESOTA.—By Professor J. F. DOWNEY: Advanced calculus (first semester), three hours; Curve tracing (second semester), three hours.—By Professor G. N. BAUER: Determinants and solid analytic geometry, three hours.—By Dr. J. E. MANCHESTER: Differential equations, three hours.—By Professor F. P. LEAVENWORTH: Method of least squares (first semester), three hours.—By Professor H. T. EDDY: Fourier's series and spherical harmonics (first semester), three hours; Theoretical mechanics, three hours.

UNIVERSITY OF NEBRASKA.—By Professor W. E. DAVIS: Number theory, three hours (first semester); Finite differences, three hours (second semester).—By Professor CANDY: Analytic geometry, three hours; Differential equations, three hours.—By Professor MORITZ: Mathematical pedagogy, three hours.—By Mr. H. A. MORRISON: Algebra, three hours.

NORTHWESTERN UNIVERSITY. By Professor H. S. WHITE: Algebraic twisted curves.—By Dr. J. W. YOUNG: Differential equations of physics.

UNIVERSITY OF PENNSYLVANIA. — By Professor E. S. CRAWLEY: Advanced methods of plane analytic geometry, two hours; Theory of numbers, three hours. — By Professor G. E. FISHER: Differential equations, two hours; Theory of functions of a complex variable (first half year), three hours; Elliptic functions (second half year), three hours. — By Professor I. J. SCHWATT: Theory of functions of a real variable, three hours. — By Dr. G. H. HALLETT: Theory of surfaces, two hours; Theory of groups, two hours. — By Dr. B. S. EASTON: Theory of higher equations, two hours; Algebra (Netto's book will be read in German), one hour. — By Dr. F. H. SAFFORD: Fourier's series and spherical harmonics, three hours. — By Dr. L. I. NEIKIRK: Galois theory of equations, two hours; Introduction to the theory of modular functions, two hours.

PRINCETON UNIVERSITY. — By Professor H. B. FINE: Elements of the theory of functions; Theory of functions of a complex variable; Ordinary differential equations. — By Professor H. D. THOMPSON: Theory of curves and surfaces; Advanced infinitesimal geometry; Special topics in metrical geometry. — By Professor E. O. LOVETT: Partial differential equations (second semester); Lie's transformation groups and applications; Analytic mechanics. — By Mr. A. H. WILSON: Elliptic functions. — By Dr. GILLESPIE: Theory of substitutions and of invariants. — By Dr. L. P. EISENHART: Deformation of surfaces (second semester); Congruences and complexes (second half year). Each course is given two hours a week.

STANFORD UNIVERSITY. — By Professor R. L. GREEN: Theory of equations, three hours; Modern analytic geometry, two hours. — By Professor G. A. MILLER: Elementary theory of groups, three hours; Projective geometry, two hours (first semester); Theory of numbers, two hours (second semester). — By Professor H. F. BLICHFELDT: Differential equations, three hours. — By Professor L. M. HOSKINS: Theoretical mechanics, five hours.

SYRACUSE UNIVERSITY. — By Professor W. H. METZLER: Advanced analytic geometry (first half year) three hours; Modern geometry (second half year), three hours; Elliptic

integrals and elliptic functions, three hours; General theory of functions of a complex variable, three hours. — By Professor E. D. ROE: Symmetric functions and invariants, three hours; Advanced calculus and differential equations, three hours; Analytic mechanics, three hours; Theory of equations, three hours; Analytic trigonometry (first half year), one hour; Determinants (second half year), one hour. — By Professor W. G. BULLARD: Projective geometry and higher plane curves, three hours.

UNIVERSITY OF VIRGINIA. — By Professor W. H. ECHOLS: Theory of functions, three hours; Geometry, three hours. — By Professor J. M. PAGE: Differential equations, three hours; Differential geometry, three hours. — By Mr. W. B. STONE: Advanced differential and integral calculus, three hours.

UNIVERSITY OF WISCONSIN. — By Professor C. A. VAN VELZER: Differential equations, three hours; Theory of numbers, two hours. — By Professor C. S. SLICHTER: Theoretical mechanics, three hours; Newtonian potential function, two hours; Theoretical hydrodynamics, two hours. — By Professor E. B. SKINNER: Quaternions, three hours; Differential geometry (first semester), three hours; Theory of invariants, two hours; Theory of groups of a finite order, three hours. — By Professor L. W. DOWLING: Projective geometry, two hours; Theory of functions, two hours; Advanced projective geometry, two hours.

(Summer session.) — By Professor C. S. SLICHTER: Differential equations; Analytic mechanics. — By Professor L. W. DOWLING: Invariants; Theory of functions. Each course will be given five hours a week.

PROFESSOR J. SOBOTKA, of the technical high school at Brünn has been appointed professor of mathematics at the Bohemian university at Prague.

LORD KELVIN was unanimously elected to the chancellorship of the University of Glasgow, April 6.

PROFESSOR C. GUICHARD has been elected a member of the Paris academy of sciences in the section of geometry; Professor V. VOLTERRA has been elected corresponding member of the academy.

DR. E. FISCHER has been appointed docent in the technical high school at Brünn.

AT the installation of President C. S. Howe at the Case School of Applied Science on May 11, the AMERICAN MATHEMATICAL SOCIETY was officially represented by Professor C. A. WALDO.

AT the University of Michigan Professor ALEXANDER ZIWET has been promoted to a full professorship in mathematics, and Professor J. L. MARKLEY to a junior professorship.

PROFESSOR MAXIME BÔCHER has been made full professor of mathematics at Harvard University.

DR. G. A. BLISS, of Chicago, has been appointed assistant professor of mathematics in the University of Missouri.

DR. J. E. MANCHESTER has been appointed instructor in mathematics at the University of Minnesota.

DR. L. C. KARPINSKI, of the Oswego Normal School, will have charge of the mathematical courses of New York University during the summer session of 1904.

DR. C. N. HASKINS has been appointed instructor in mathematics at Cornell University.

MR. H. A. MORRISON has been appointed assistant in mathematics at the University of Nebraska. Messrs. C. H. SISAM and O. P. AKERS have been appointed assistants in mathematics at Cornell University.

MR. E. A. MILLER, of the Massachusetts Institute of Technology, has been granted a leave of absence and will spend next year at European universities.

RECENT catalogues of second-hand books on mathematics: H. WELTER, 4 rue Bernard-Palissy, Paris, catalogue No. 112, 34 pages (about 1300 titles), mathematics, physics and chemistry; Mayer and Müller, Prinz Louis Ferdinandstrasse 2, Berlin N. W., catalogue No. 203, pp. 130, mathematics; Gustav

Fock, Schlossgasse 7–9, Leipzig, catalogue No. 240, 1726 titles on mathematics; Süddeutsches Antiquariat, Galleriestrasse 20, Munich, catalogue No. 55, 1231 titles in mathematics, physics and astronomy.

CHARLTON T. LEWIS, eminent publicist, died at Morristown, N. J., on May 26th, at the age of seventy. Dr. Lewis was a member of the Actuarial Society of America and, since 1891, of the AMERICAN MATHEMATICAL SOCIETY.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMES (L. D.). On a theorem of analysis situs relating to the division of the plane or of space by a closed curve or surface. (*Bulletin of the American Mathematical Society*, (2) Vol. 10, pp. 301–305.)

BACH (H.). Die Umformungen der Kettenbrüche. (Diss.) Giessen, 1903. 8vo. 40 pp.

BASTIAN (A.). Das logische Rechnen und seine Aufgaben. Berlin, 1903. 8vo. 176 pp. M. 4.00

BJERKNES (V.). Karl Anton Bjerknes. Gedächtnisrede, gehalten vor der Gesellschaft der Wissenschaften zu Christiania am 17. April 1903. Mit einem Bildnis. Leipzig, Barth, 1904. 8vo. 31 pp. M. 1.20

BRAHY (E.). Exercices méthodiques de calcul intégral. Nouvelle édition entièrement revue. Paris, Gauthier-Villars, 1903. 8vo. 301 pp. Fr. 5.00

CAPELLI (A.). Lezioni sulla teoria delle forme algebriche. (Paris, Gauthier-Villars), Napoli, Pellerano, 1902. 8vo. 295 pp. (Lith.). Fr. 10.00

ENGEL (F.). See GRASSMANN (H.).

FESTSCHRIFT, Ludwig Boltzmann gewidmet zum 60sten Geburtstage, 20. II. 1904. Leipzig, Barth, 1904. 8vo. 12 + 930 pp., 2 plates, 1 portrait. M. 18.00

GAZZANIGA (P.). Gli elementi della teoria dei numeri. Verona e Padova, Drucker, 1903. 8vo. 7 + 408 pp. Fr. 8.00

GRASSMANN (H.). Gesammelte mathematische und physikalische Werke. Auf Veranlassung der mathematisch-physischen Klasse der kgl. Sächsischen Gesellschaft der Wissenschaften und unter Mitwirkung der Herren J. Lüroth, E. Study, J. Grassmann, H. Grassmann d. J., G. Scheffers herausgegeben von F. Engel. Vol. II, Teil 1: Die Abhandlungen zur Geometrie und Analysis, herausgegeben von E. Study, G. Scheffers und F. Engel. Leipzig, Teubner, 1904. 8vo. 460 pp.

HALSTED (G. B.). The message of non-Euclidean geometry. (*Science*, new series, Vol. 19, pp. 401–413.) 8vo. 30 pp.

HELLER (S.). Untersuchungen über die natürlichen Gleichungen krummer Flächen. (Diss.) Kiel, 1904, 8vo. 65 pp.

HERTZ (W.). Ueber partielle Differentialgleichungen, die in der Variationsrechnung vorkommen. (Diss., Kiel.) Göttingen, Kästner, 1903. 4to. 56 pp.

LAPLACE (P. S.). Oeuvres complètes, publiées sous les auspices de l'Académie des sciences par les secrétaires perpétuels, avec le concours de Puiseux, F. Tisserand, J. Hoüel, Souillart, H. Poincaré et Lebeuf. Nouvelle édition avec un beau portrait de Laplace, gravé sur cuivre par Tony Goutière. (1878–1904.) Vol. 13: Mémoires extraits de la *Connaissance des temps*. Paris, Gauthier-Villars, 1904. 4to. Fr. 15.00

LEBESGUE (H.). Leçons sur l'intégration et la recherche des fonctions primitives. Paris, Gauthier-Villars, 1904. 8vo. 7 + 139 pp. Fr. 3.50

LIETZMANN (W.). Ueber das biquadratische Reziprozitätsgesetz in algebraischen Zahlkörpern. (Diss.) Göttingen, 1904. 8vo. 4 + 94 pp.

MADE (H.). Ueber Fareysche Doppelreihen. (Diss.) Giessen, 1903. 8vo. 33 pp. 3 plates.

SANCHEZ RAMOS (E.). Notas relativas á los poliedros estrellados y de Arquimedes. Paris, Gauthier-Villars, 1903. 8vo. 1 plate. Fr. 1.00

SCHEFFERS (G.). See GRASSMANN (H.).

SIMON (O.). Ueber mehrdimensionale Geometrie. (Progr.) Ungarisch-Hradisch, 1903. 4to. 24 pp.

SPORER (B.). Niedere Analysis. 2te, verbesserte Auflage, 2ter Abdruck. Leipzig, Göschen, 1903. 16mo. 179 pp. (Sammlung Göschen, No. 53.) Cloth. M. 0.80

STUDY (E.). See GRASSMANN (H.).

THYBAUT (A.). See TRESSE (A.).

TRESSE (A.) et THYBAUT (A.). Cours de géométrie analytique, à l'usage des candidats à l'Ecole centrale des arts et manufactures, aux écoles des mines, à l'Ecole des ponts et chaussées et des élèves de première année de mathématiques spéciales. Paris, Colin, 1904. 8vo. 3 + 256 pp. Fr. 6.00

II. ELEMENTARY MATHEMATICS.

ALEU (M. L.). Pizarras de trigonometría rectilinea. Madrid, 1903. 4to. 41 pp. Fr. 3.00

BAKER (W. M.) and BOURNE (A. A.). Elementary geometry, book 4. London, Bell, 1904. 12mo. Cloth. 1s.

BOURNE (A. A.) See BAKER (W. M.).

BRIGGS (W.) and BRYAN (G. H.). Tutorial algebra (advanced course). 2d edition. (University Tutorial Series.) London, Clive, 1904. 12mo. 620 pp. Buckram. 6s. 6d.

BRYAN (G. H.). See BRIGGS (W.).

COMBEROUSSE (C. DE). Géométrie élémentaire, plane et dans l'espace. 4e édition, revue et augmentée. Paris, Gauthier-Villars, 1904. 8vo. Fr. 8.00

DONADT (A.). See LÜBSEN (H. B.).

GRÉVY (A.). Eléments d'algèbre, à l'usage des élèves des classes de troisième A à première A et B (programmes du 31 mai 1902). 4e édition. Paris, Vuibert et Nony, 1904. 16mo. 196 pp. Cloth. Fr. 1.75

GUGLIELMI (A.). Elementi di algebra e complementi di aritmetica razionale per i licei. Napoli, Romano, 1904. 16mo. 8 + 285 pp. Fr. 2.20

HALL (H. S.) and STEVENS (F. H.). A school geometry. Parts 3 and 4, containing the substance of Euclid, books 2 and 3 and part of book 4. London and New York, Macmillan, 1904. 12mo. 7 + 240 pp. Cloth. $0.35

HARTWIG (T.). Schule der Mathematik zum Selbstunterrichte. (Beilage zu "Das Wissen für Alle.") Vol. I: Algebra. Wien, Perles, 1904. 8vo. 3 + 280 pp. M. 3.50

HOČEVAR (F.). Lehrbuch der Geometrie nebst einer Sammlung von Uebungsaufgaben, für Obergymnasien. 5te, verbesserte Auflage. Wien, Tempsky, 1904. 8vo. 4 + 298 pp. Cloth. M. 3.70

KLEINSCHMIDT (E.). Geometrische Anschauungslehre für die ersten vier Klassen an Mädchen-Lyzeen. (In neuer Rechtschreibung.) Wien, Hölder, 1903. 8vo. 4 + 103 pp. Cloth. M. 1.65

——. Leitfaden der Geometrie und des geometrischen Zeichnens für Mädchen-Bürgerschulen. Teil II: 2te Klasse. 3te Auflage; inhaltlich unveränderter, nach der neuen Rechtschreibung hergestellter Abdruck der 2ten Auflage. Wien, Hölder, 1904. 8vo. 3 + 60 pp. Cloth. M. 0.80

LONDON University matriculation mathematics papers. Examination papers set from Jan. 1891 to Jan. 1904. (University Tutorial Series.) London, Clive, 1904. 12mo. 1s. 6d.

LÜBSEN (H. B.). Ausführliches Lehrbuch der ebenen und sphärischen Trigonometrie. Zum Selbstunterricht mit Rücksicht auf die Zwecke des praktischen Lebens bearbeitet. 18te Auflage, neu bearbeitet von A. Donadt. Leipzig, Brandstetter, 1904. 8vo. 5 + 146 pp. M. 2.40

PAPELIER (G.). Formulaire de mathématiques spéciales (algèbre; trigonométrie; géométrie analytique). Paris, Vuibert et Nony, 1904. 8vo. 224 pp.

PICKWORTH (C. N.). Logarithms for beginners. London, Whittaker, 1904. 12mo. Limp leather. 1s.

PRESSLAND (A. J.). Introduction to the study of geometry. London, Rivington, 1904. 12mo. 40 pp. Cloth. 1s.

SCHRAM (J.). Leibnizens Definitionen der Ebene und der Geraden. 8vo. 22 pp.

STEVENS (F. H.). See HALL (H. S.).

WARREN (A. T). Experimental and theoretical course in geometry. 2d edition. London, Frowde, 1904. 12mo. 268 pp. Cloth. 2s.

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III. APPLIED MATHEMATICS.

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